Modified Hennings invariants

- Hennings 3-mnf invariants
- Modified trace

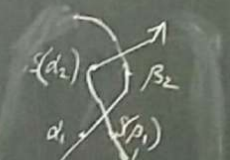
Blanchet + Geer

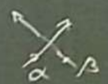
- WOW - Theorem

Blanchet + Gainutdinov

(H, r, R) ribbon Hopf algebra

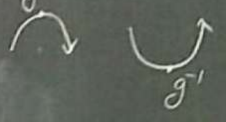
$J_T \in H^{\otimes m}$ $\quad T$ m-component tangle

$C :=$ 



$R = \sum \alpha \otimes \beta$

$$J_C = \beta_2 \alpha_1 \otimes S(\beta_1) S(\alpha_2) \in \left(H^{\otimes 2}\right)^H$$



$$\boxed{S^2(x) = g x g^{-1}, \quad \Delta g = g \otimes g, \quad S(g) = g^{-1}}$$

Properties of g pivotal

$$J_T(V_1, \dots, V_m) = \left(tr_g^{V_1} \otimes \dots \otimes tr_g^{V_m}\right) J_T \qquad tr_g^V(x) = trace^V(gx)$$

· WOW - Theorem } Blanchet + Garnatdinov

"Kirby color" right integral

$\mu \in H^{*}$

$$\boxed{(\mu \otimes id)\Delta x = \mu(x)\mathbb{1} \quad \forall x \in H}$$

Thm (Radford, Sweedler) μ exists and unique up scalar for any f.d. Hopf algebra

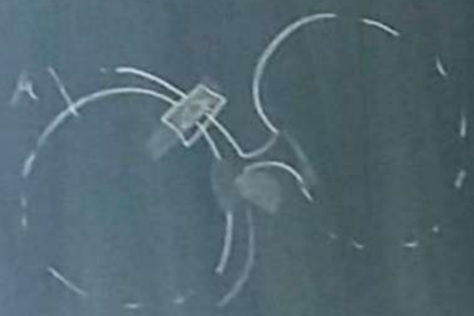
H uni-modular $\left(\exists\, c \in H,\ xc = \varepsilon(x)c = cx \ \ \forall x \in H \quad P_1^* = P_1 \right)$

$$\mu \in \underset{\overset{\cup}{tr_q^V}}{qChar(H)} = \{ f \in H^* \mid f(xy) = f(S^2(y)x) \}$$

$\mu($

<u>Claim</u>: $\mu^{\otimes m}(J_T)$ is invariant under KZ-move

<u>Proof</u>:



$y)x)$

$\mu(r^{\pm 1}) \neq 0 \Leftarrow H$ factorizable

- $\exists$ non-degen Hopf pairing
- $M = R_{21} R_{12} = \sum c_i \otimes c_i'$

$\{c_i\}_{i \in I}$, $\{c_i'\}$ are bases of H.

- Drinfeld map is isomorph

$\mathcal{D} : H^* \to H$

$f \longmapsto (f \otimes id) M$

$\mathcal{D}|_{qChar} : qChar(H) \longrightarrow Z(H)$

$V \longmapsto J(\;_V)$

Thm (Hennings) Let (H, R, r) f.d. factorizable Hopf algebra

$M = S^3(L)$ $\quad b_\pm$ # $\substack{pos \\ neg}$ eigenval lk(L

$$\mathcal{H}(M) = \frac{\mu^{\otimes m}(J_T)}{\mu(r)^{b_+} \, \mu(r^{-1})^{b_-}}$$

<u>Remark:</u> $\forall z \in Z(H)$ $S(z) = z$

$(1 \otimes z) \Delta z = z \otimes z$

$\mu \mapsto \mu_z$ another Kirby color

$\mu_z(x) := \mu(zx)$

$\exists z$ s.t. $\mu_z = \sum_{i=0}^{r-1} \operatorname{qdim} V_i \; \mathrm{tr}_q^{V_i}$

$q^{2r} = 1$

<u>Kerler</u> $\mathcal{H}(S^2 \times S^1) \neq 0 \iff$ H-mod is semi-simple

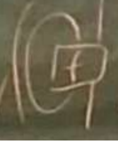
right + left

Modified trace: family of linear functions

H-pmod = f.d projective H-modules

$$\{ t_V : \mathrm{End}(V) \to \mathbb{K} \}_{V \in H\text{-pmod}}$$

cyclicity: $t_V(fg) = t_W(gf)$

partial trace property $t_{V \otimes W}\, f = t_V($  $) = t_W($  $)$

Restricted quantum $\mathfrak{sl}(2)$ $\mathcal{U}$

$E, K, F \qquad q^{2r} = 1 \qquad E^i F^j K^p$

$EK = q^{-2} KE$

$E^r = F^r = 0, \; K^{2r} = 1$

H-mod

$V_0 \quad V_1 \quad V_2 \quad \cdots \quad V_{r-1} \quad V_r =$ simple projective

$u \subset \mathcal{U} \qquad t_u$

Restricted quantum $sl(2)$ $\mathcal{U}$

$E, K, F \qquad q^{2r} = 1 \qquad E^i F^j K^p$

$EK = q^{-2} KE \qquad \mathcal{U}^{\otimes 2} \ni M, r \in \mathcal{U}$

$$\boxed{E^r = F^r = 0, \; K^{2r} = 1}$$

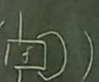
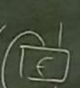
H-mod

$V_0 \quad V_1 \quad V_2 \quad \cdots \quad V_{r-1} \quad V_r =$ simple projective

$\mathcal{U} \circlearrowright \mathcal{U} \qquad t_{\mathcal{U}} \qquad \boxed{\mathrm{End}_{\mathcal{U}}\, \mathcal{U} = \mathcal{U}^{op}}$

$HH_0(\mathcal{U}) = \mathcal{U}/[\mathcal{U},\mathcal{U}] \qquad \mathrm{Char}(\mathcal{U}) = \{ f \in \mathcal{U}^* \mid f(xy) = f(yx) \}$

Thm (Blanchet-Geer) $\exists\, \mathrm{Tr}' = t_{\mathcal{U}} \in \mathrm{Char}(\mathcal{U})$

$\mathrm{Tr}'($  $) = \mathrm{Tr}'($  $) \quad \forall f \in \mathrm{End}_{\mathcal{U}}\, \mathcal{U}^{\otimes 2}$

$\langle\, ,\, \rangle : Z(H) \times HH_0(\mathcal{U}) \longrightarrow K$

$(z, h) \longmapsto \mathrm{Tr}'(zh)$

non-degenerate.

$\langle\, ,\, \rangle : (\mathcal{U}^{\otimes m})^{\mathcal{U}} \times HH_0(\mathcal{U}^{\otimes m}) \longrightarrow K$

$\langle x, y \rangle := t_{\mathcal{U}^{\otimes m}}(l_x r_y)$

Th

f.

M

Rema

$(1 \otimes$

$\mu_2($

Lemma: Let T is a m-component string link

$$J_T \subset \left(\mathcal{U}^{\otimes m}\right)^{\mathcal{U}}$$

Invariant: $M = S^3(L^0) \supset L = (L^+, L^-)$ with L^0 having m_0 components, L^+ having m^+ components labelled z^+, L^- having m^- components labelled h^-

$$z^+ \in \left(Z(\mathcal{U})\right)^{\otimes m^+}$$

$$h^- \in \left(HH_0(\mathcal{U})\right)^{\otimes m^-}$$

$T = (T^+, T^0, T^-)$ string link

$$\mathcal{H}^{\log}(M, L) = \delta^s \left\langle \left(\mu_z^{\otimes m^+} \otimes \mu^{\otimes m_0} \otimes id\right) J_T, h^- \right\rangle$$

top. invar. of 3-mnf.

$\boxed{\text{Sw-1}}$ $\mathrm{Char}(\mathcal{U})^{\circ} \underset{\mathrm{tr}=\mu_g}{\overset{\mu}{\xleftarrow{\ \ \simeq\ \ }}} q\mathrm{Char}(\mathcal{U})$

$$f_g \longleftarrow f$$

$$f \longmapsto f_{g^{-1}}$$

$$f_g(xy) = f(gxy) = f(S^2(g)gx) =$$

$$= f(gyx) = f_g(yx)$$

<u>WoW-Theorem.</u> (B-Blanchet + Geer-Patureau-Mirand)

For any uni-modular pivotal Hopf algebra (H, g) $\exists!$ (up to scalar) right modified trace on H-pmod. Moreover, $t_H(f) = \mu_g(f(1))$

$\forall f \in \mathrm{End}_H H$

If in addition, $a = g^2 \iff$ left modif trace = right modif trace

$$(\mu \otimes \mathrm{id})\Delta x = \mu(x)1$$

$$(\mathrm{id} \otimes \mu)\Delta x = \mu(x)\underline{a}$$