The Kashiwara-Vergne Problem and Topology

Handout, video, and links at ω/

Abstract. I will describe the general "expansions" machine whose inputs are topics in topology (and more) and whose outputs are problems in algebra. There are many inputs the machine can take, and many outputs it produces, but I will concentrate on just one input/output pair. When fed with a certain class of knotted 2-dimensional objects in 4-dimensional space, it outputs the Kashiwara-Vergne Problem (1978 ω/KV, solved Alekseev-Meinrenken 2006 ω/AM, elucidated Alekseev-Torossian 2008-2012 ω/AT), a problem about convolutions on Lie groups and Lie algebras.

The Kashiwara-Vergne Conjecture. There exist two series F and G in the completed free Lie algebra FL in generators x and y so that

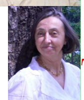
$$x+y-\log e^y e^x = (1-e^{-\operatorname{ad} x})F + (e^{\operatorname{ad} y}-1)G \quad \text{in } FL$$

and so that with $z = \log e^x e^y$,

$$\operatorname{tr}(\operatorname{ad} x)\partial_x F + \operatorname{tr}(\operatorname{ad} y)\partial_y G \quad \text{in cyclic words}$$
$$= \frac{1}{2}\operatorname{tr}\left(\frac{\operatorname{ad} x}{e^{\operatorname{ad} x}-1} + \frac{\operatorname{ad} y}{e^{\operatorname{ad} y}-1} - \frac{\operatorname{ad} z}{e^{\operatorname{ad} z}-1} - 1\right)$$

Implies the loosely-stated convolutions statement: Convolutions of invariant functions on a Lie group agree with convolutions of invariant functions on its Lie algebra.


Kashiwara


Vergne


Alekseev


Meinrenken


Torossian

The Machine. Let G be a group, $\mathcal{K} = \mathbb{Q}G = \{\sum a_i g_i \colon a_i \in \mathbb{Q},\ g_i \in G\}$ its group-ring, $\mathcal{I} = \{\sum a_i g_i \colon \sum a_i = 0\} \subset \mathcal{K}$ its augmentation ideal. Let

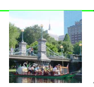
$$\mathcal{A} = \operatorname{gr}\mathcal{K} := \widehat{\bigoplus_{m\geq 0}} \mathcal{I}^m/\mathcal{I}^{m+1}.$$

P.S. $(\mathcal{K}/\mathcal{I}^{m+1})^*$ is Vassiliev / finite-type / polynomial invariants.

Note that $\mathcal{A}$ inherits a product from G.

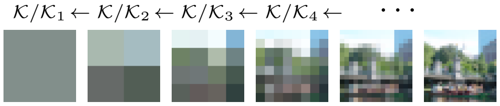
Definition. A linear $Z\colon \mathcal{K} \to \mathcal{A}$ is an "expansion" if for any $\gamma \in \mathcal{I}^m$, $Z(\gamma) = (0, \ldots, 0, \gamma/\mathcal{I}^{m+1}, *, \ldots)$, and a "homomorphic expansion" if in addition it preserves the product.

Example. Let $\mathcal{K} = C^\infty(\mathbb{R}^n)$ and $\mathcal{I} = \{f \colon f(0) = 0\}$. Then $\mathcal{I}^m = \{f \colon f \text{ vanishes like } |x|^m\}$ so $\mathcal{I}^m/\mathcal{I}^{m+1}$ is degree m homogeneous polynomials and $\mathcal{A} = \{\text{power series}\}$. The Taylor series is a homomorphic expansion!

Just for fun. $\mathcal{K} = \left\{ \right.$  $\left. \right\} = \left(\text{The set of all 2D projections of reality } (= \mathbb{Q}^3\mathbb{R}^2)\right)$

$\mathcal{K}/\mathcal{K}_1 \leftarrow \mathcal{K}/\mathcal{K}_2 \leftarrow \mathcal{K}/\mathcal{K}_3 \leftarrow \mathcal{K}/\mathcal{K}_4 \leftarrow \cdots$

Rotate
Colour Correct
Adjoin



An expansion Z is a choice of a "progressive scan" algorithm.

Rotate
Colour Correct
Adjoin

$\mathcal{K}/\mathcal{K}_1 \oplus \mathcal{K}_1/\mathcal{K}_2 \oplus \mathcal{K}_2/\mathcal{K}_3 \oplus \mathcal{K}_3/\mathcal{K}_4 \oplus \mathcal{K}_4/\mathcal{K}_5 \oplus \mathcal{K}_5/\mathcal{K}_6 \oplus \cdots$

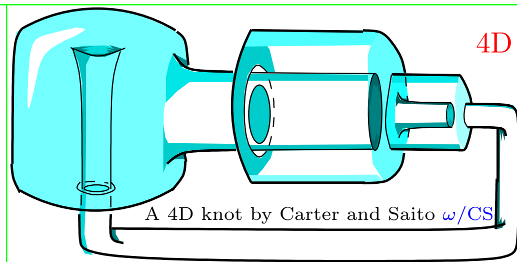
$\mathcal{K}/\mathcal{K}_1 = \mathbb{R}^3$, $\quad \mathcal{K}_3/\mathcal{K}_4 = \ker(\mathcal{K}/\mathcal{K}_4 \to \mathcal{K}/\mathcal{K}_3)$

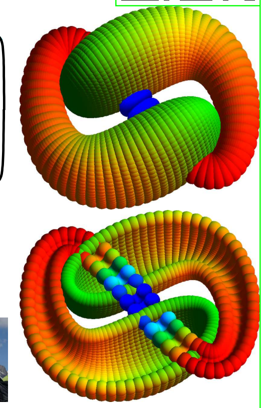
In the finitely presented case, finding Z amounts to solving a system of equations in a graded space.

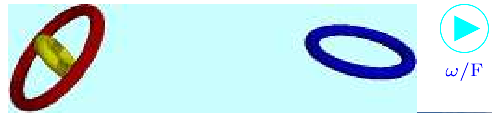
Theorem (with Zsuzsanna Dancso, ω/WKO). There is a bijection between the set of homomorphic expansions for $w\mathcal{K}$ and the set of solutions of the Kashiwara-Vergne problem. This is the tip of a major iceberg!


Dancso, ω/ZD

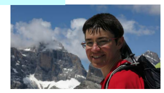
4D Knots.


A 4D knot by Carter and Saito ω/CS

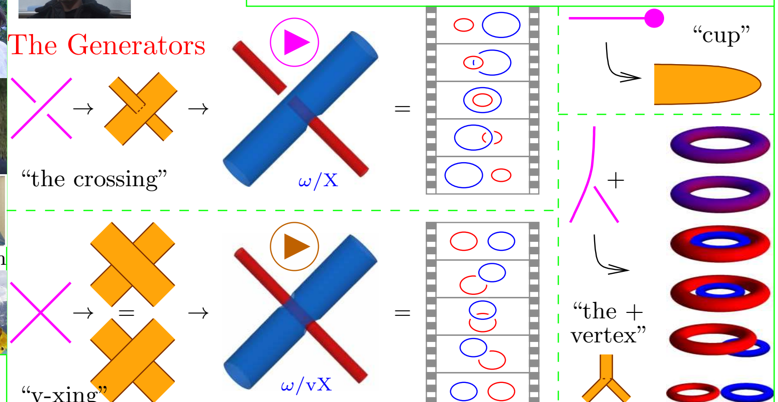



ω/F

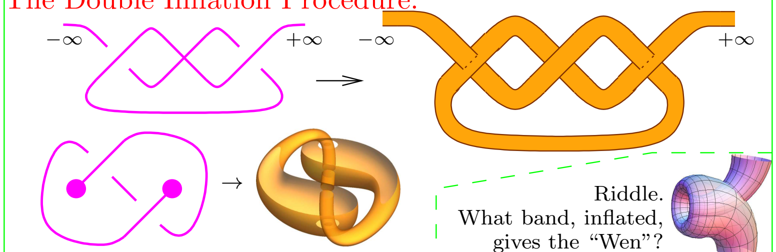

Satoh

Dalvit ω/Dal


The Generators

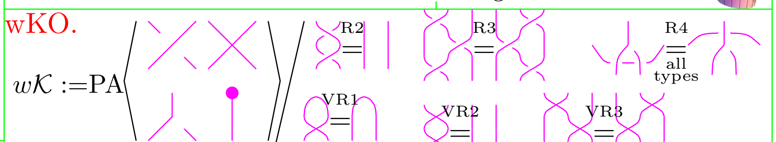


The Double Inflation Procedure.

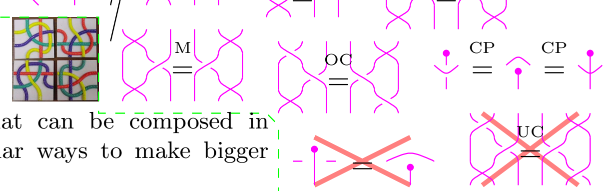


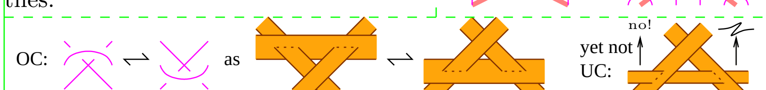
Riddle. What band, inflated, gives the "Wen"?

wKO.

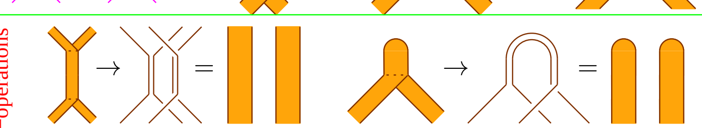


"Planar Algebra": The objects are "tiles" that can be composed in arbitrary planar ways to make bigger tiles.





The unary w-operations

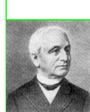


The Machine generalizes to arbitrary algebraic structures!

$$\begin{tikzcd}
 & u\mathcal{K} \arrow[rr, "Z\colon \text{the Kontsevich}", "\text{integral}"'] \arrow[dl] \arrow[dd] & & \mathcal{A}^u \arrow[dr] \arrow[dd] & & \ker_{\mathfrak{g}} \arrow[dl] \\
v\mathcal{K} \arrow[rrrr, "Z\colon \text{hard work}", "\text{Etingof-Kazhdan}"'] \arrow[dr] & & & & \mathcal{A}^v \arrow[dl] \arrow[dr] & (\text{given } \mathfrak{g}) \\
 & w\mathcal{K} \arrow[rr, "Z\colon \text{today's work}"] & & \mathcal{A}^w & & \text{Around } \operatorname{Rep}(\mathfrak{g})
\end{tikzcd}$$

Why care? and more!

Five year plan: polynomial-time knot polynomials!



"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

www.katlas.org

Dror Bar-Natan: Talks: Northeastern-1602:

Help Needed!

I wish I could read, or maybe I wish I could write, a survery paper on Taylor expansions for groups with an abstract and a "master table" as below. Oh wait, I'm actually in the process of writing that paper⋆! But it's going hard because I'm underqualified — I don't know the existing general material well enough, and many of the entries of the "master table" require specialized knowledge that I don't have.

Abstract. First year students learn that the Taylor expansion Z_T carries functions into power series, and that it has some nice algebraic properties (e.g. multiplicativity, $Z_T(fg) = Z_T(f)Z_T(g)$). It is less well known that the same game can be played within arbitrary groups: there is a natural way to say "a Taylor expansion Z for elements of an arbitrary group G", and a natural way to carry the algebraic properties of the Taylor expansion to this more general context. In the case of a general G "Taylor expansions" (expansions with the same good properties as Z_T) may or may not exist, may or may not be unique, may or may not separate group elements, and a further good property which is hidden in the case of Z_T, "quadraticity", may or may not hold.

The purpose of this expository note is to properly define all the notions in the above paragraph, to enumerate some classes of groups whose theory of expansions we either understand or wish to understand, to indicate the relationship between these notions and the notions of "finite type invariants" and "unipotent" and "Mal'cev" completions, and to point out (with references) that our generalization of "expansions" to arbitrary groups is merely the tip of an iceberg, for almost everything we say can be generalized further to "expansions for arbitrary algebraic structures".

⋆See http://drorbn.net/ExQu. Also see Suciu-Wang, arXiv:1504.08294.

✔:=Yes, ✘:=No, ?:=Unknown (to the author).
Superscripts: see "table footnotes" below.

Group(s) G	Faithful Z?	Taylor Z?	Quadratic?	See
1. Finite / torsion groups	✘[1]	✔[2]	✔[2]	Sec. 2.1
2. Free Abelian groups $\mathbb{Z}^n$	✔	✔	✔	Sec. 1.4.2
3. Free groups FG_n	✔	✔	✔	Sec. 1.4.3
4. LOT and LOF groups	✘[3]	✔	✔	Sec. 4.1
5. Knot and pure tangle groups	✘[3]	✔	✔	Sec. 4.2
6. Link groups	✘[4]	✔	✔	Sec. 4.2
7. Pure braid groups PB_n	✔	✔	✔	Sec. 1.5
8. Reduced free groups RF_n	✔	✔	✘	Sec. 4.3
9. Reduced (homotopy) pure braid groups RPB_n	✔	✔	✘	
10. Pure v-braid groups PvB_n	?	✘	✔	
11. Pure w-braid groups PwB_n	✔	✔	✔	
12. Pure f-braid groups PfB_n				Merkov
13. Annular braids				
14. Elliptic pure braid groups PB_n^1 (braids on the torus)	?	✔	✘	
15. Higher genus pure braid groups $PB_n^{>1}$ (braids on high genus surfaces)	?	?	✘	arXiv:math/0309245?
16. Commutators $[G,G]$ in general				
17. Commutators $[PuB_n, PuB_n]$				
18. Commutators $[PvB_n, PvB_n]$				
19. Commutators $[PwB_n, PwB_n]$				
20. Hilden braids				
21. Mexican plait braids				Kurpita-Murasugi
22. Cactus groups				
23. Fundamental groups of surfaces		✔	✔	
24. Mapping class groups				
25. Torelli groups				Hain
26. Right-angled Artin groups		✔	✔	
27. General Artin groups				
28. Groups from BEER				arXiv:math/0509661
29. Groups from Brochier				arXiv:1209.0417
30. Poly-free groups				arXiv:math/0603470

Table 1: Some groups and their expansion properties.

Table footnotes. 1. Except $G = \{e\}$. 2. In an empty manner. 3. Except $G = FG_n$. 4. Except $G = \mathbb{Z}^n$.