The Kashiwara-Vergne Problem and Topology

Handout, video, and links at ω/

Abstract. I will describe the general "expansions" machine whose inputs are topics in topology (and more) and whose outputs are problems in algebra. There are many inputs the machine can take, and many outputs it produces, but I will concentrate on just one input/output pair. When fed with a certain class of knotted 2-dimensional objects in 4-dimensional space, it outputs the Kashiwara-Vergne Problem (1978 ω/KV, solved Alekseev-Meinrenken 2006 ω/AM, elucidated Alekseev-Torossian 2008-2012 ω/AT), a problem about convolutions on Lie groups and Lie algebras.

The Kashiwara-Vergne Conjecture. There exist two series F and G in the completed free Lie algebra FL in generators x and y so that

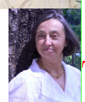
$$x+y-\log e^y e^x = (1-e^{-\operatorname{ad} x})F+(e^{\operatorname{ad} y}-1)G \quad \text{in } FL$$

and so that with $z=\log e^x e^y$,

$$\operatorname{tr}(\operatorname{ad} x)\partial_x F+\operatorname{tr}(\operatorname{ad} y)\partial_y G \quad \text{in cyclic words}$$
$$=\frac{1}{2}\operatorname{tr}\left(\frac{\operatorname{ad} x}{e^{\operatorname{ad} x}-1}+\frac{\operatorname{ad} y}{e^{\operatorname{ad} y}-1}-\frac{\operatorname{ad} z}{e^{\operatorname{ad} z}-1}-1\right)$$

Implies the loosely-stated convolutions statement: Convolutions of invariant functions on a Lie group agree with convolutions of invariant functions on its Lie algebra.


Kashiwara


Vergne


Alekseev


Meinrenken


Torossian

The Machine. Let G be a group, $\mathcal{K}=\mathbb{Q}G=\{\sum a_i g_i\colon a_i\in\mathbb{Q},\ g_i\in G\}$ its group-ring, $\mathcal{I}=\{\sum a_i g_i\colon \sum a_i=0\}\subset\mathcal{K}$ its augmentation ideal. Let

$$\mathcal{A}=\operatorname{gr}\mathcal{K}:=\widehat{\bigoplus_{m\geq 0}}\mathcal{I}^m/\mathcal{I}^{m+1}.$$

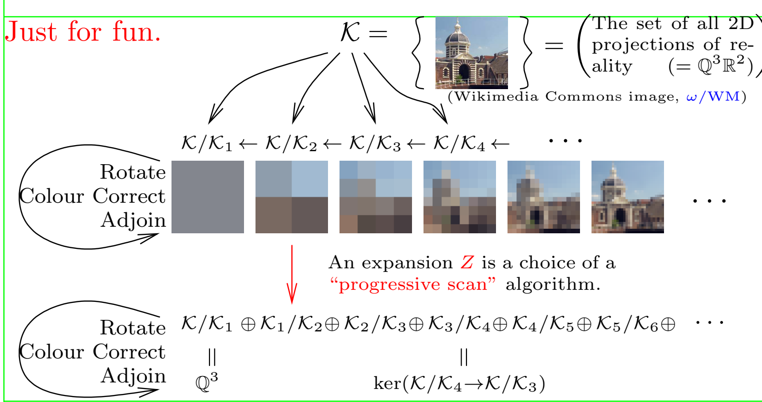
P.S. $(\mathcal{K}/\mathcal{I}^{m+1})^*$ is Vassiliev / finite-type / polynomial invariants.

Note that $\mathcal{A}$ inherits a product from G.

Definition. A linear $Z\colon\mathcal{K}\to\mathcal{A}$ is an "expansion" if for any $\gamma\in\mathcal{I}^m$, $Z(\gamma)=(0,\dots,0,\gamma/\mathcal{I}^{m+1},*,\dots)$, and a "homomorphic expansion" if in addition it preserves the product.

Example. Let $\mathcal{K}=C^\infty(\mathbb{R}^n)$ and $\mathcal{I}=\{f\colon f(0)=0\}$. Then $\mathcal{I}^m=\{f\colon f \text{ vanishes like } |x|^m\}$ so $\mathcal{I}^m/\mathcal{I}^{m+1}$ is degree m homogeneous polynomials and $\mathcal{A}=\{\text{power series}\}$. The Taylor series is a homomorphic expansion!

Just for fun.

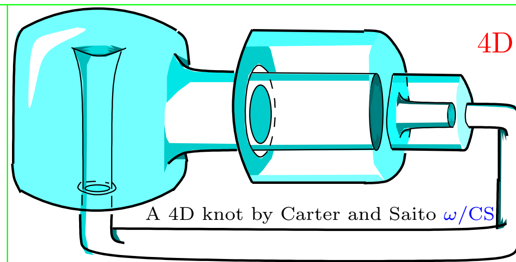


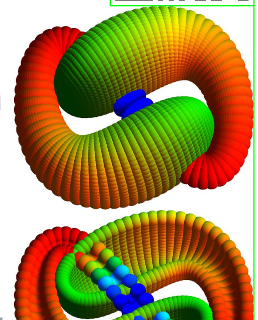
In the finitely presented case, finding Z amounts to solving a system of equations in a graded space.

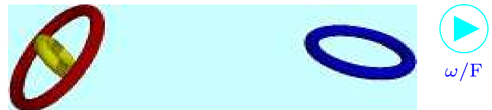
Theorem (with Zsuzsanna Dancso, ω/WKO). There is a bijection between the set of homomorphic expansions for $w\mathcal{K}$ and the set of solutions of the Kashiwara-Vergne problem. This is the tip of a major iceberg!


Dancso, ω/ZD

4D Knots.


A 4D knot by Carter and Saito ω/CS

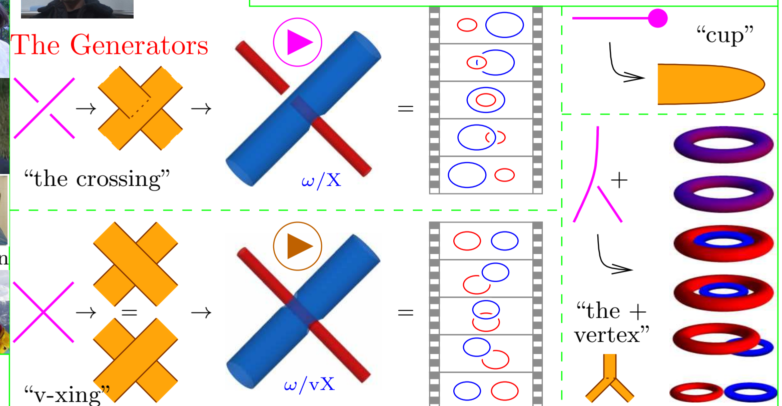



ω/F

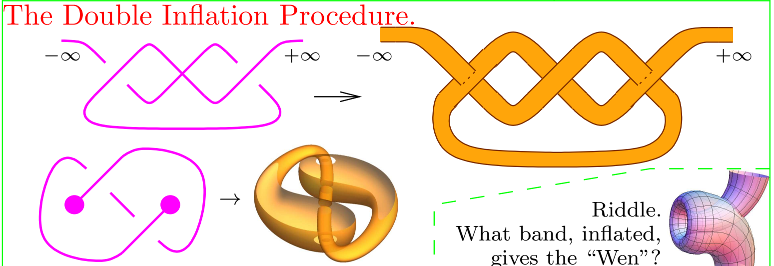

Satoh


Dalvit ω/Dal

The Generators

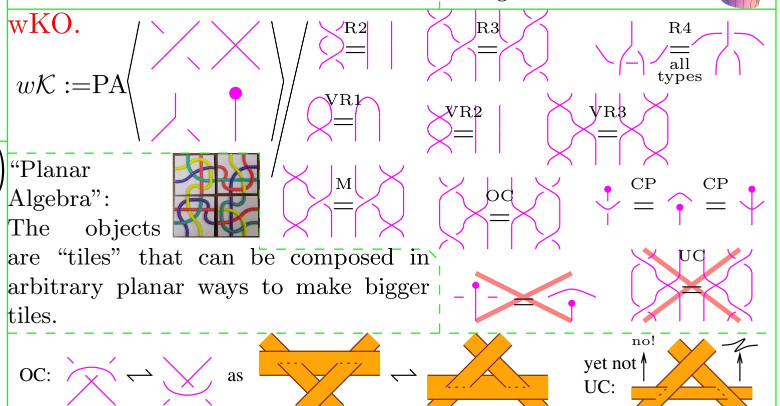


The Double Inflation Procedure.



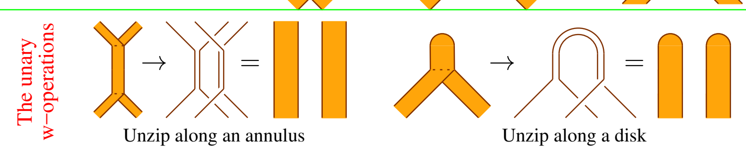
Riddle. What band, inflated, gives the "Wen"?

wKO.

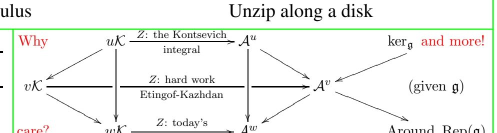


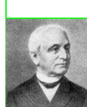
"Planar Algebra": The objects are "tiles" that can be composed in arbitrary planar ways to make bigger tiles.

The unary w-operations



The Machine generalizes to arbitrary algebraic structures!





"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

www.katlas.org The Knot Atlas Anyone Can Edit