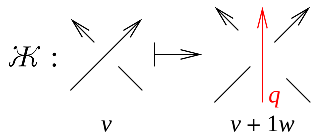
Crossing the Crossings

Abstract. The subject will be very close to Manturov's representation of vB_n into $\mathrm{Aut}(FG_{n+1})$ — I'll describe how I think about it in terms of a very simple minded map $\mathcal{K}$ from n-component v-tangles to $(n+1)$-component w-tangles. It is possible that you all know this already. Possibly my talk will be very short — it will be as long as it is necessary to describe $\mathcal{K}$ and say a few more words, and if this is little, so be it.

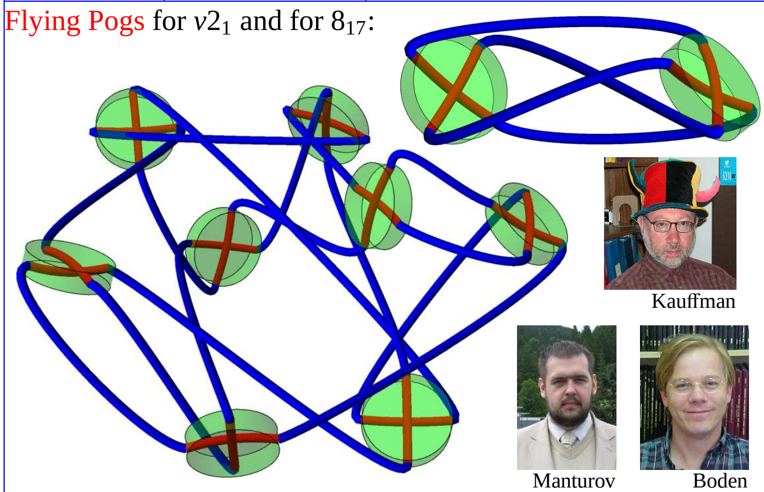
All you need is $\mathcal{K}$... • What is its domain? • What is its target? • Why should one care?



Virtual Knots. Virtual knots are the algebraic structure underlying the Reidemeister presentation of ordinary knots, without the topology. Locally they are knot diagrams modulo the Reidemeister relations; globally, who cares? So,

$$vT = \mathrm{CA}\langle \nearrow\!\!\nwarrow, \nwarrow\!\!\nearrow : R1, R2, R3\rangle \qquad \mathrm{CA} = \text{“Circuit Algebra”}$$

Flying Pogs for $v2_1$ and for 8_{17}:

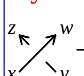


No! Note that also (with PA = "Planar Algebra")

$$vT = \mathrm{PA}\langle \nearrow\!\!\nwarrow, \nwarrow\!\!\nearrow, \times : R1, R2, R3, VR1, VR2, VR3, M\rangle,$$

but I have a prejudice, or a deeply held belief, that this is morally wrong!

My moment of reckoning. Manturov's $VG(K)$: [Ma, BGHNW]



$$z = xyx^{-1},\ w = x \qquad z = x^{-1}yx,\ w = x \qquad z = q^{-1}yq,\ w = qxq^{-1}$$

Manturov's μ: $vB_n \to \mathrm{Aut}(F(x_1, \dots, x_n, q))$: [Ma, BGHNW]

$$\sigma_i = \nearrow\!\!\nwarrow_i \mapsto \begin{cases} x_i \mapsto x_i x_{i+1} x_i^{-1} \\ x_{i+1} \mapsto x_i \end{cases} \qquad \tau_i = \times_i \mapsto \begin{cases} x_i \mapsto q x_{i+1} q^{-1} \\ x_{i+1} \mapsto q^{-1} x_i q \end{cases}.$$

Easy resolution. Setting $y_i := q^i x_i q^{-i}$, we find that μ is equivalent to

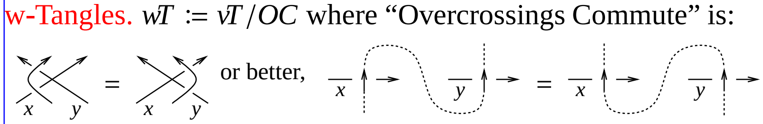
$$\nearrow\!\!\nwarrow_i \mapsto \begin{cases} y_i \mapsto y_i q^{-1} y_{i+1} q y_i^{-1} \\ y_{i+1} \mapsto q y_i q^{-1} \end{cases} \qquad \times_i \mapsto \begin{cases} y_i \mapsto y_{i+1} \\ y_{i+1} \mapsto y_i \end{cases},$$

and to me, virtual braids are anyways always pure. So really,

$$\sigma_{ij} \mapsto \begin{cases} y_i \mapsto q y_i q^{-1} \\ y_j \mapsto y_i^{-1} q^{-1} y_j q y_i \end{cases}.$$

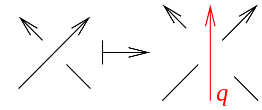
But why does it exist? Especially, wherefore $vB_n \to wB_{n+1}$?

w-Tangles. $wT := vT/OC$ where "Overcrossings Commute" is:



π_1 is defined on wT; Artin's representation ϕ is defined on wB_n.

Back to $\mathcal{K}$. The "crossing the crossings" map $\mathcal{K}: vT_n \to wT_{n+1}$ is defined by the picture below. Equally well, it is $\mathcal{K}: vB_n \to wB_{n+1}$. Better, it is $\mathcal{K}: vT_n \to (nv + 1w)T$ or $\mathcal{K}: vB_n \to (nv + 1w)B$.



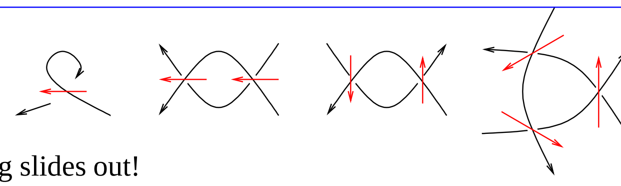
Claims.

1. $\mathcal{K}$ is well defined.
2. On u-links, $\mathcal{K}$ "factors".
3. $\mathcal{K}$ does not respect OC.
4. $\mathcal{K}$ recovers Manturov's VG and μ: $VG(K) = \pi_1(\mathcal{K}(K))$, $\mu = \mathcal{K} \circ \phi = \phi /\!\!/ \mathcal{K}$.

Even better, $\mathcal{K}$ pulls back *any* invariant of 2-component w-knots to an invariant of virtual knots. in particular, there is a wheel-valued "non-commutative" invariant ω as in [BN] and DBN: Talks: Hamilton-1412 (next page).

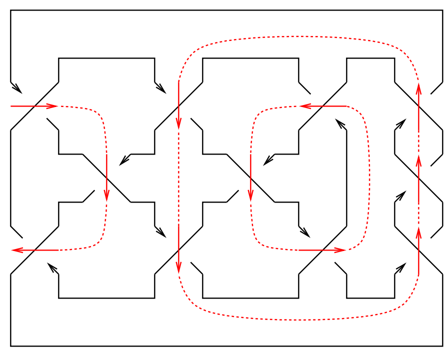
Likely, the various "2-variable Alexander polynomials" for virtual knots arise in this way.

Proof of 1.



Everything slides out!

Proof of 2. The net "red flow" into every face is 0, so the red arrows can be paired. They form cycles that can hover off the picture.



No proof of 3. Well, there simply is no proof that OC is respected, and it's easy to come up with counter-examples.

Proof of 4. A simple verification, except my conventions are off...

References.

[BN] D. Bar-Natan, *Balloons and Hoops and their Universal Finite Type Invariant, BF Theory, and an Ultimate Alexander Invariant,* Acta Mathematica Vietnamica **40-2** (2015) 271–329, arXiv:1308.1721.

[BGHNW] H. U. Boden, A. I. Gaudreau, E. Harper, A. J. Nicas, and L. White, *Virtual Knot Groups and Almost Classical Knots,* arXiv:1506.01726.

[Ma] V. O. Manturov, *On Invariants of Virtual Links,* Acta Applicandae Mathematica **72-3** (2002) 295–309.

Prejudices should always be re-evaluated!

Tangles, Wheels, Balloons

Abstract. I will describe a computable, non-commutative invariant of tangles with values in wheels, almost generalize it to some balloons, and then tell you why I care. Spoilers: tangles are you know what, wheels are linear combinations of cyclic words in some alphabet, balloons are 2-knots, and one reason I care is because quantum field theory predicts more than I can actually get (but also less).

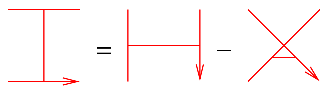
Why I like "non-commutative"? With $FA(x_i)$ the free associative non-commutative algebra,

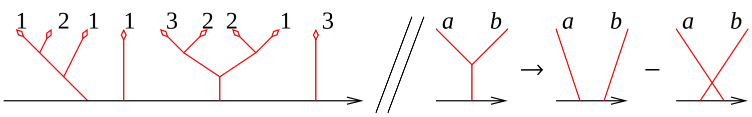
$$\dim \mathbb{Q}[x,y]_d \sim d \ll 2^d \sim \dim FA(x,y)_d.$$

Why I like "computable"?

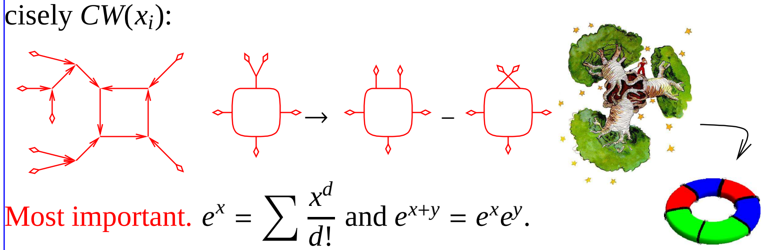
- Because I'm weird.
- Note that π_1 isn't computable.

Preliminaries from Algebra. $FL(x_i)$ denotes the free Lie algebra in (x_i); $FL(x_i)$ = (binary trees with AS vertices and coloured leafs)/(IHX relations). There an obvious map $FA(FL(x_i)) \to FA(x_i)$ defined by $[a,b] \to ab - ba$, which in itself, is IHX.



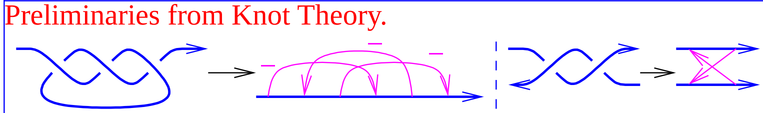


$CW(x_i)$ denotes the vector space of cyclic words in (x_i): $CW(x_i) = FA(x_i)/(x_i w = w x_i)$. There an obvious map $CW(FL(x_i)) \to CW(x_i)$. In fact, connected uni-trivalent 2-in-1-out graphs with univalents with colours in $\{1,\ldots,n\}$, modulo AS and IHX, is precisely $CW(x_i)$:

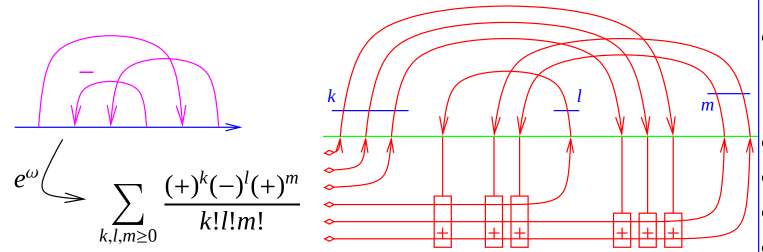


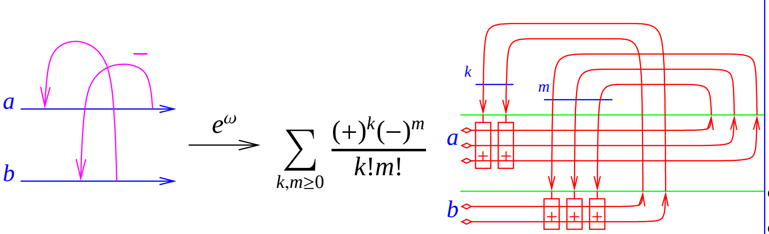
Most important. $e^x = \sum \frac{x^d}{d!}$ and $e^{x+y} = e^x e^y$.

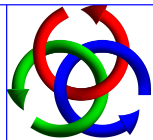
Preliminaries from Knot Theory.



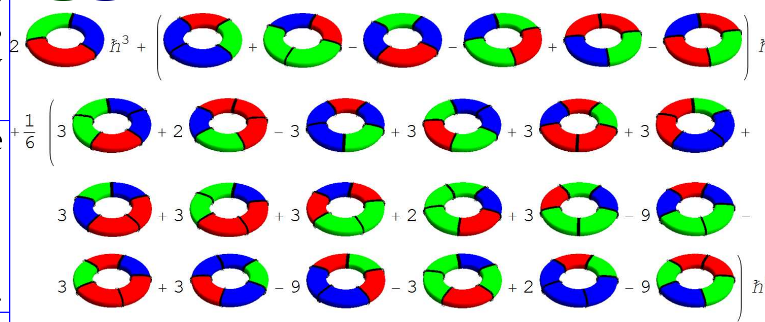
Theorem. ω, the connected part of the procedure below, is an invariant of S-component tangles with values in $CW(S)$:





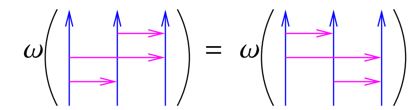


ω is practically computable! For the Borromean tangle, to degree 5, the result is: (see [BN])

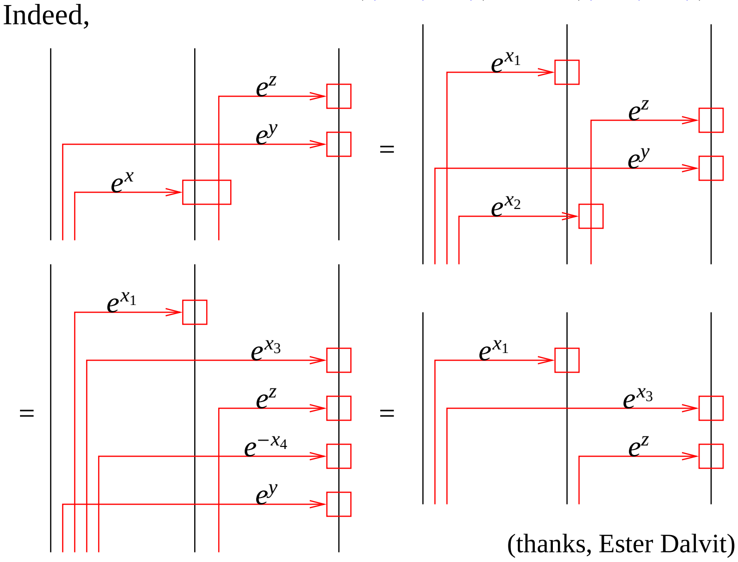


Proof of Invariance.

Need to show:



Indeed,



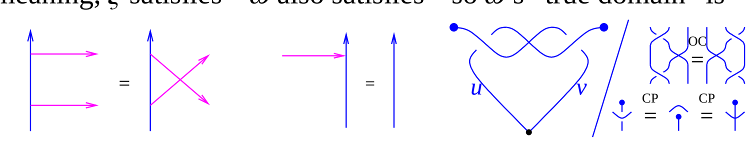
(thanks, Ester Dalvit)

Further Facts

- ω is really the second part of a (trees,wheels)-valued invariant $\zeta = (\lambda, \omega)$. The tree part λ is just a repackaging of the Milnor μ-invariants.
- On u-tangles, ζ is equivalent to the trees&wheels part of the Kontsevich integral, except it is computable and is defined with no need for a choice of parenthesization.
- On long/round u-knots, ω is equivalent to the Alexander polynomial.
- The multivariable Alexander polynomial (and Levine's factorization thereof [Le]) is contained in the Abelianization of ζ [BNS].
- ω vanishes on braids.
- Related to / extends Farber's [Fa]?
- Should be summed and categorified.
- Extends to v and descends to w:

Does ω extend to balloons?

meaning, ζ satisfies ω also satisfies so ω's "true domain" is



- Agrees with BN-Dancso [BND1, BND2] and with [BN].
- ζ, ω are universal finite type invariants.
- Using $\bowtie$: $v\mathcal{K}_n \to w\mathcal{K}_{n+1}$, defines a strong invariant of v-tangles / long v-knots. ($\bowtie$ in LaTeX: ωεβ/zhe)