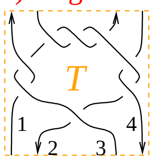
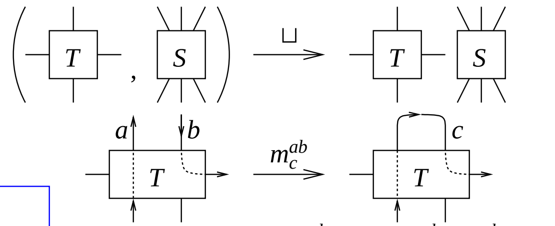
Work in Progress on

Polynomial Time Knot Polynomials, A

Abstrant. The value of things is inversely correlated with their computational complexity. "Real time" machines, such as our brains, only run linear time algorithms, and there's still a lot we don't know. Anything we learn about things doable in linear time is truly valuable. Polynomial time we can in-practice run, even if we have to wait; these things are still valuable. Exponential time we can play with, but just a little, and exponential things must be beautiful or philosophically compelling to deserve attention. Values further diminish and the aesthetic-or-philosophical bar further rises as we go further slower, or un-computable, or ZFC-style intrinsically infinite, or large-cardinalish, or beyond.
I will explain some things I know about polynomial time knot polynomials and explain where there's more, within reach.

(v-)Tangles.



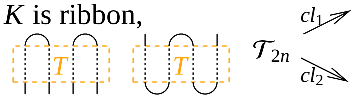


(meta-associativity: $m_a^{ab}/\!/m_a^{ac} = m_b^{bc}/\!/m_a^{ab}$)

Why Tangles?

- Finitely presented.
- Divide and conquer proofs and computations.
- "Algebraic Knot Theory": If K is ribbon, $z(K) \in \{cl_2(\zeta)\colon cl_1(\zeta)=1\}$.

(Genus and crossing number are also definable properties).



Faster is better, leaner is meaner!

Theorem 1. ∃! an invariant z_0: {pure framed S-component tangles} $\to \Gamma_0(S) := R \times M_{S\times S}(R)$, where $R = R_S = \mathbb{Z}((T_a)_{a\in S})$ is the ring of rational functions in S variables, intertwining

$$\left(\begin{array}{c|c}\omega_1 & S_1\\ \hline S_1 & A_1\end{array},\begin{array}{c|c}\omega_2 & S_2\\ \hline S_2 & A_2\end{array}\right) \xrightarrow{\sqcup} \begin{array}{c|cc}\omega_1\omega_2 & S_1 & S_2\\ \hline S_1 & A_1 & 0\\ S_2 & 0 & A_2\end{array},$$

$$\begin{array}{c|ccc}\omega & a & b & S\\ \hline a & \alpha & \beta & \theta\\ b & \gamma & \delta & \epsilon\\ S & \phi & \psi & \Xi\end{array} \xrightarrow[\substack{T_a,T_b\to T_c\\ \mu:=1-\beta}]{m_c^{ab}} \left(\begin{array}{c|cc}\mu\omega & c & S\\ \hline c & \gamma+\alpha\delta/\mu & \epsilon+\delta\theta/\mu\\ S & \phi+\alpha\psi/\mu & \Xi+\psi\theta/\mu\end{array}\right),$$

and satisfying $\left(\uparrow_a;\ {}_a\nearrow_b,\ {}_b\nwarrow_a\right) \xrightarrow{z_0} \left(\begin{array}{c|c}1 & a\\ \hline a & 1\end{array};\ \begin{array}{c|cc}1 & a & b\\ \hline a & 1 & 1-T_a^{\pm1}\\ b & 0 & T_a^{\pm1}\end{array}\right)$.

In Addition • The matrix part is just a stitching formula for Burau/Gassner [LD, KLW, CT].

- $K \mapsto \omega$ is Alexander, mod units.
- $L \mapsto (\omega, A) \mapsto \omega\det'(A-I)/(1-T')$ is the MVA, mod units.
- The "fastest" Alexander algorithm.
- There are also formulas for strand deletion, reversal, and doubling.
- Every step along the computation is the invariant of something.
- Extends to and more naturally defined on v/w-tangles.
- Fits in one column, including propaganda & implementation.

Implementation key idea: ωεβ/Demo

$$(\omega, A = (\alpha_{ab})) \leftrightarrow (\omega, \lambda = \textstyle\sum \alpha_{ab} t_a h_b)$$

```
ΓCollect[Γ[ω_, λ_]] := Γ[Simplify[ω],
   Collect[λ, h_, Collect[#, t_, Factor] &]];
Format[Γ[ω_, λ_]] := Module[{S, M},
   S = Union@Cases[Γ[ω, λ], (h | t)a_ :> a, ∞];
   M = Outer[Factor[∂h#1t#2 λ] &, S, S];
   M = Prepend[M, t# & /@ S] // Transpose;
   M = Prepend[M, Prepend[h# & /@ S, ω]];
   M // MatrixForm];
```

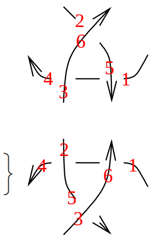
```
Γ /: Γ[ω1_, λ1_] Γ[ω2_, λ2_] := Γ[ω1 * ω2, λ1 + λ2];
ma_b_→c_[Γ[ω_, λ_]] := Module[{α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ, μ},
  (α β θ; γ δ ϵ; ϕ ψ Ξ) = (∂ta,ha λ  ∂ta,hb λ  ∂ta λ;
                           ∂tb,ha λ  ∂tb,hb λ  ∂tb λ;
                           ∂ha λ     ∂hb λ     λ) /. (t | h)a|b → 0;
  Γ[(μ = 1 - β) ω, {tc, 1}.(γ + α δ / μ   ϵ + δ θ / μ;
                            ϕ + α ψ / μ   Ξ + ψ θ / μ).{hc, 1}]
   /. {Ta → Tc, Tb → Tc} // ΓCollect];
Rpa_b_ := Γ[1, {ta, tb}.(1  1 - Ta; 0  Ta).{ha, hb}];
Rma_b_ := Rpab /. Ta → 1 / Ta;
```

Meta-Associativity **Runs.**

```
ζ = Γ[ω, {t1, t2, t3, tS}.(α11 α12 α13 θ1;
                            α21 α22 α23 θ2;
                            α31 α32 α33 θ3;
                            ϕ1  ϕ2  ϕ3  Ξ).{h1, h2, h3, hS}];
(ζ // m12→1 // m13→1) == (ζ // m23→2 // m12→1)
```

```
True
```

R3 ... divide and conquer!

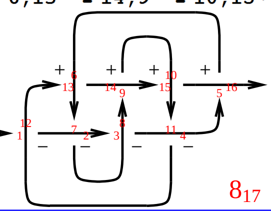



```
{Rm51 Rm62 Rp34 // m14→1 // m25→2 // m36→3,
  Rp61 Rm24 Rm35 // m14→1 // m25→2 // m36→3}
```

$$\left\{\begin{pmatrix} 1 & h_1 & h_2 & h_3\\ t_1 & \frac{T_3}{T_2} & 0 & 0\\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0\\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1\end{pmatrix}, \begin{pmatrix} 1 & h_1 & h_2 & h_3\\ t_1 & \frac{T_3}{T_2} & 0 & 0\\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0\\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1\end{pmatrix}\right\}$$

```
z = Rm12,1 Rm27 Rm83 Rm4,11 Rp16,5 Rp6,13 Rp14,9 Rp10,15;
Do[z = z // m1k→1, {k, 2, 16}];
z
```

$$\begin{pmatrix} 11 - \frac{1}{T_1^3} + \frac{4}{T_1^2} - \frac{8}{T_1} - 8\,T_1 + 4\,T_1^2 - T_1^3 & h_1\\ t_1 & 1\end{pmatrix}$$



8_{17}

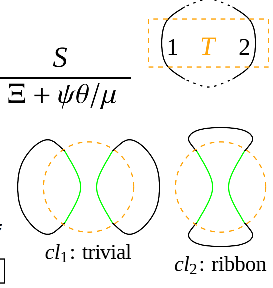
Closed Components. The Halacheva trace satisfies $m_c^{ab}/\!/\operatorname{tr}_c = m_c^{ba}/\!/\operatorname{tr}_c$ and computes the MVA for all links in the atlas, but its domain is not understood:

$$\begin{array}{c|cc}\omega & c & S\\ \hline c & \alpha & \theta\\ S & \psi & \Xi\end{array} \xrightarrow[\mu:=1-\alpha]{\operatorname{tr}_c} \begin{array}{c|c}\mu\omega & S\\ \hline S & \Xi+\psi\theta/\mu\end{array}$$

Halacheva

```
trc_[Γ[ω_, λ_]] := Module[{α, θ, ψ, Ξ},
   (α θ; ψ Ξ) = (∂tc,hc λ  ∂tc λ; ∂hc λ  λ) /. (t | h)c → 0;
   Γ[ω (1 - α), Ξ + ψ * θ / (1 - α)] // ΓCollect];
(ζ // m12→1 // tr1) == (ζ // m21→1 // tr1)
```

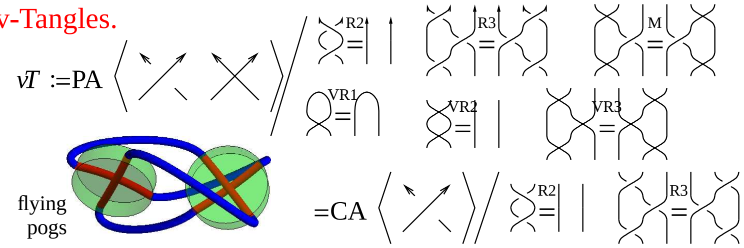
```
True
```



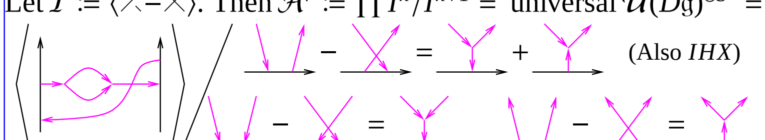
Weaknesses. • m_c^{ab} and tr_c are non-linear. • The product ωA is always Laurent, but my current proof takes induction with exponentially many conditions. • I still don't understand tr_c. • I still don't understand "unitarity".

Where does it come from?

v-Tangles.



Let $\mathcal{I} := \langle\overcrossing - \undercrossing\rangle$. Then $\mathcal{A}^v := \prod \mathcal{I}^n/\mathcal{I}^{n+1}$ ="universal $\mathcal{U}(D\mathfrak{g})^{\otimes S}$"=

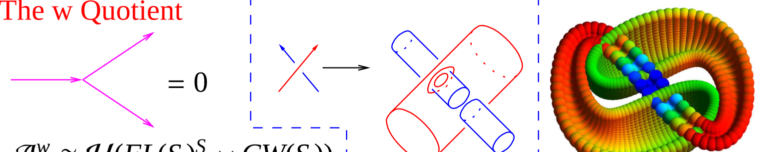


Fine print: No sources no sinks, AS vertices, internally acyclic, deg = (#vertices)/2.

Theorem. [EK, En] There exists a homomorphic expansion (universal finite type invariant) $Z\colon v\mathcal{T} \to \mathcal{A}^v$.

Too hard! Let's look for "meta-monoid" quotients.

The w Quotient

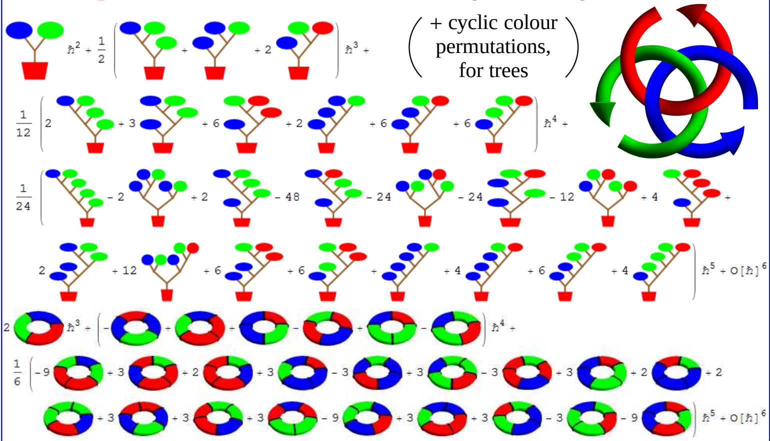


$$\mathcal{A}^w \cong \mathcal{U}(FL(S)^S \ltimes CW(S))$$

Polynomial Time Knot Polynomials, B

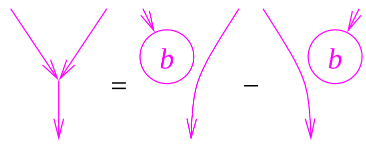
Theorem 2 [BND]. ∃! a homomorphic expansion, aka a homomorphic universal finite type invariant Z^w of pure w-tangles. $z^w := \log Z^w$ takes values in $FL(S)^S \times CW(S)$.

z is computable. z of the Borromean tangle, to degree 5 [BN]:



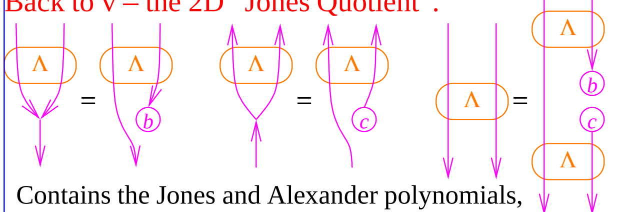
(I have a fancy free-Lie calculator!) Nice, but too hard!

Proposition [BN]. Modulo all relations that universally hold for the 2D non-Abelian Lie algebra and after some changes-of-variable, z^w reduces to z_0.



$[u, v] = b_u v - b_v u$

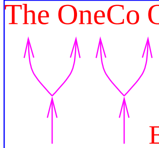
Back to v – the 2D "Jones Quotient".



V. Jones

Contains the Jones and Alexander polynomials, still too hard!

The OneCo Quotient.



= 0, only one co-bracket is allowed.

Everything should work, and everything is being worked!

The Abstract Context

Definition. (Compare [BNS, BN]) A meta-monoid is a functor M: (finite sets, injections)→(sets) (think "$M(S)$ is quantum G^S", for G a group) along with natural operations $*: M(S_1) \times M(S_2) \to M(S_1 \sqcup S_2)$ whenever $S_1 \cap S_2 = \emptyset$ and $m^{ab}_c : M(S) \to M((S \setminus \{a, b\}) \sqcup \{c\})$ whenever $a \neq b \in S$ and $c \notin S \setminus \{a, b\}$, such that

$$\text{meta-associativity:}\quad m^{ab}_a /\!/ m^{ac}_a = m^{bc}_b /\!/ m^{ab}_a$$

$$\text{meta-locality:}\quad m^{ab}_c /\!/ m^{de}_f = m^{de}_f /\!/ m^{ab}_c$$

and, with $\epsilon_b = M(S \hookrightarrow S \sqcup \{b\})$,

$$\text{meta-unit:}\quad \epsilon_b /\!/ m^{ab}_a = Id = \epsilon_b /\!/ m^{ba}_a.$$

Claim. Pure virtual tangles PvT form a meta-monoid.

Theorem. $S \mapsto \Gamma_0(S)$ is a meta-monoid and $z_0 : PvT \to \Gamma_0$ is a morphism of meta-monoids.

Strong Conviction. There exists an extension of Γ_0 to a bigger meta-monoid $\Gamma_{01}(S) = \Gamma_0(S) \times \Gamma_1(S)$, along with an extension of z_0 to $z_{01} : PvT \to \Gamma_{01}$, with

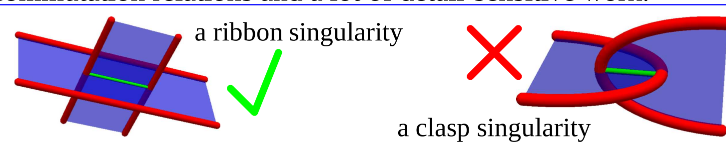
$$\Gamma_1(S) < \langle S \sqcup S \times S \sqcup S \times S \times S \sqcup S \times S \times S \times S \rangle.$$

Furthermore, upon reducing to a single variable everything is polynomial size and polynomial time.

Furthermore, Γ_{01} is given using a "meta-2-cocycle ρ^{ab}_c over Γ_0": In addition to $m^{ab}_c \to m^{ab}_{0c}$, there are R_S-linear $m^{ab}_{1c} : \Gamma_1(S \sqcup \{a, b\}) \to \Gamma_1(S \sqcup \{c\})$, a meta-right-action $\alpha^{ab} : \Gamma_1(S) \times \Gamma_0(S) \to \Gamma_1(S)$ R_S-linear in the first variable, and a first order differential operator (over R_S) $\rho^{ab}_c : \Gamma_0(S \sqcup \{a, b\}) \to \Gamma_1(S \sqcup \{c\})$ such that

$$(\zeta_0, \zeta_1) /\!/ m^{ab}_c = \left(\zeta_0 /\!/ m^{ab}_{0c}, (\zeta_1, \zeta_0) /\!/ \alpha^{ab} /\!/ m^{ab}_{1c} + \zeta_0 /\!/ \rho^{ab}_c\right)$$

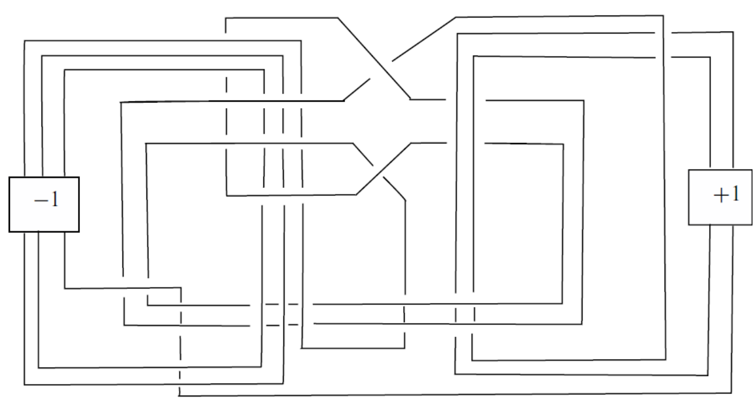
What's missing? Some commutation relations and exponentiated commutation relations and a lot of detail-sensitive work.



A bit about ribbon knots. A "ribbon knot" is a knot that can be presented as the boundary of a disk that has "ribbon singularities", but no "clasp singularities". A "slice knot" is a knot in $S^3 = \partial B^4$ which is the boundary of a non-singular disk in B^4. Every ribbon knots is clearly slice, yet,

Conjecture. Some slice knots are not ribbon.

Fox-Milnor. The Alexander polynomial of a ribbon knot is always of the form $A(t) = f(t) f(1/t)$.



[GST]: a slice knot that might not be ribbon (48 crossings).

References.

[BN] D. Bar-Natan, *Balloons and Hoops and their Universal Finite Type Invariant, BF Theory, and an Ultimate Alexander Invariant*, ωεβ/KBH, arXiv:1308.1721.

[BND] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects I-II*, ωεβ/WKO1, ωεβ/WKO2, arXiv:1405.1956, arXiv:1405.1955.

[BNS] D. Bar-Natan and S. Selmani, *Meta-Monoids, Meta-Bicrossed Products, and the Alexander Polynomial*, J. of Knot Theory and its Ramifications **22-10** (2013), arXiv:1302.5689.

[CT] D. Cimasoni and V. Turaev, *A Lagrangian Representation of Tangles*, Topology **44** (2005) 747–767, arXiv:math.GT/0406269.

[En] B. Enriquez, *A Cohomological Construction of Quantization Functors of Lie Bialgebras*, Adv. in Math. **197-2** (2005) 430-479, arXiv:math/0212325.

[EK] P. Etingof and D. Kazhdan, *Quantization of Lie Bialgebras, I*, Selecta Mathematica **2** (1996) 1–41, arXiv:q-alg/9506005.

[GST] R. E. Gompf, M. Scharlemann, and A. Thompson, *Fibered Knots and Potential Counterexamples to the Property 2R and Slice-Ribbon Conjectures*, Geom. and Top. **14** (2010) 2305–2347, arXiv:1103.1601.

[KLW] P. Kirk, C. Livingston, and Z. Wang, *The Gassner Representation for String Links*, Comm. Cont. Math. **3** (2001) 87–136, arXiv:math/9806035.

[LD] J. Y. Le Dimet, *Enlacements d'Intervalles et Représentation de Gassner*, Comment. Math. Helv. **67** (1992) 306–315.

Let's talk about China, America, Taiwan, economy, ecology, religion, democracy, censorship, and all else.

"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

www.katlas.org