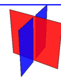
Expansions Day 5: Back to 4D

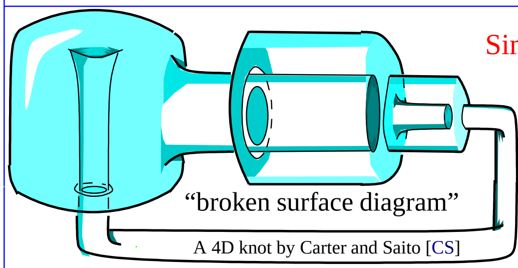
Abstract. We will repeat the 3D story of the previous 3 talks one dimension up, in 4D. Surprisingly, there's more room in 4D, and things get easier, at least when we restrict our attention to "w-knots", or to "simply-knotted 2-knots". But even then there are intricacies, and we try to go beyond simply-knotted, we are completely confused.

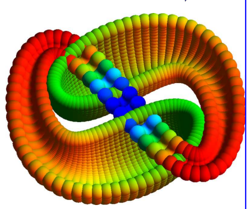
Recall.

$\mathcal{K}$ (Topology) $\xrightarrow{Z\text{: high algebra}}$ (solving finitely many equations in finitely many unknowns) $\mathcal{A} := \text{gr}\mathcal{K}$ (Combinatorics) $\xrightarrow{\text{given a "Lie" algebra } \mathfrak{g}}$ (low algebra: pictures represent formulas) "$\mathcal{U}(\mathfrak{g})$"

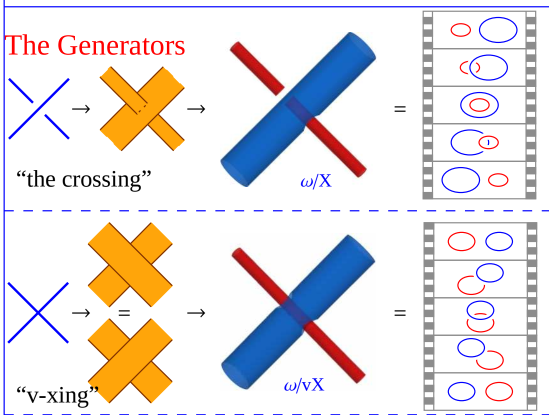
Simple 2-Knots.

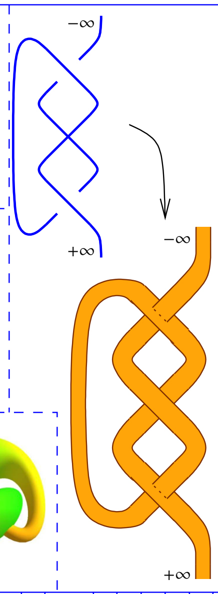


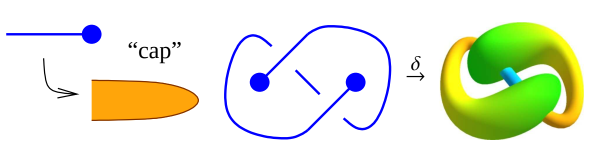




The Generators



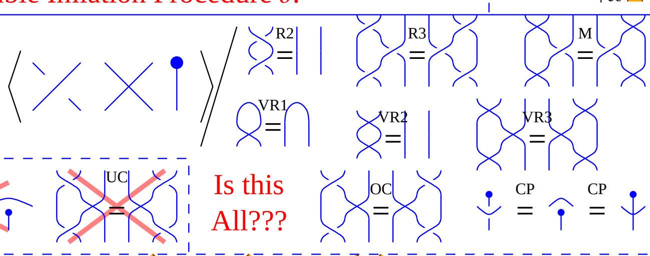




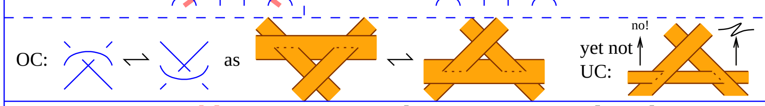
The Double Inflation Procedure δ.

w-Knots.

$w\mathcal{K}$:=PA

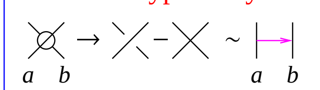


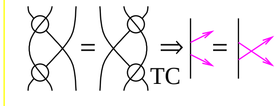
Is this All???

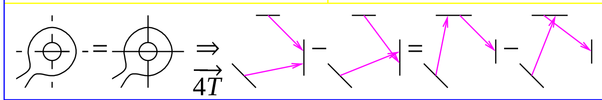


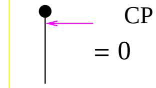
A Big Open Problem. δ maps w-knots onto simple 2-knots. To what extent is it a bijection? What other relations are required? In other words, **find a simple description of simple 2-knots**. Kawauchi [Ka] may already know the answer.

The Finite Type Story.

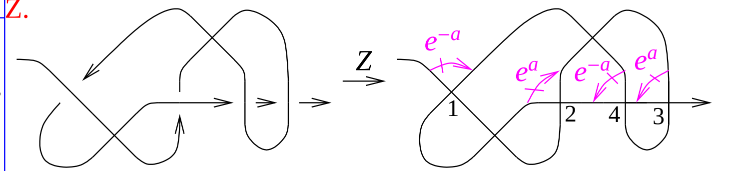




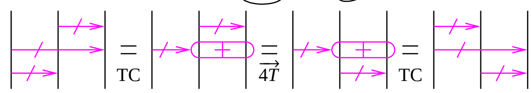




Z.

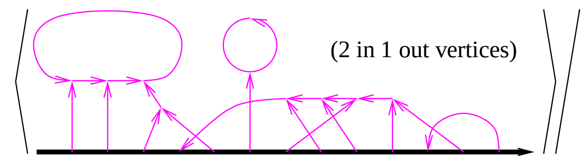


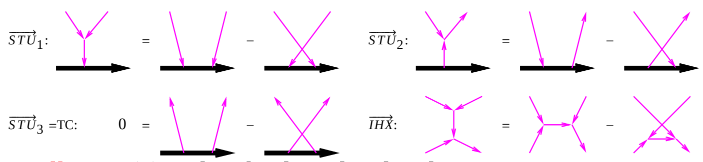
R3.



The Bracket-Rise Theorem.

$\mathcal{A}^w$ is isomorphic to





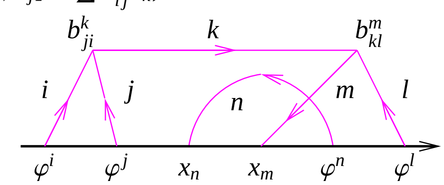
Corollaries. (1) Only wheels and isolated arrows persist:

$$\mathcal{A}^w(\uparrow_n) \cong \mathcal{U}(FL(n)^n_{tb} \ltimes CW(n)) \quad \text{and} \quad \zeta := \log Z \in FL(n)^n \times CW(n)$$

has completely explicit formulas using natural FL/CW operations [BN].
(2) Related to f.d. Lie algebras!

Low Algebra.

With (x_i) and (φ^j) dual bases of $\mathfrak{g}$ and $\mathfrak{g}^*$ and with $[x_i, x_j] = \sum b^k_{ij} x_k$, we have $\mathcal{A}^w \to \mathcal{U}$ via



Penrose

Cvitanovic

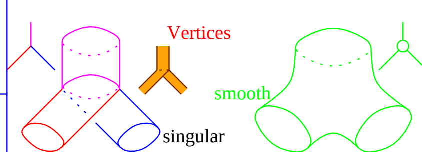
$$\longrightarrow \sum_{i,j,k,l,m,n=1}^{\dim \mathfrak{g}} b^k_{ij} b^m_{kl} \varphi^i \varphi^j x_n x_m \varphi^l \in \mathcal{U}(I\mathfrak{g} := \mathfrak{g}^* \rtimes \mathfrak{g})$$

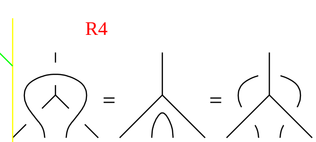
Differential Ops.

We can also interpret $\hat{\mathcal{U}}(I\mathfrak{g})$ as tangential differential operators on Fun($\mathfrak{g}$): $\varphi \in \mathfrak{g}^*$ becomes a multiplication operator, and $x \in \mathfrak{g}$ becomes a tangential derivation, in the direction of the action of ad x: $(x\varphi)(y) := \varphi([x, y])$.

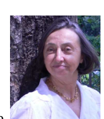
Too easy so far!

Yet once you add "foam vertices", it gets related to the Kashiwara-Vergne problem [KV] as told by Alekseev-Torossian [AT]:



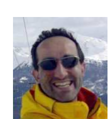


Kashiwara



Vergne

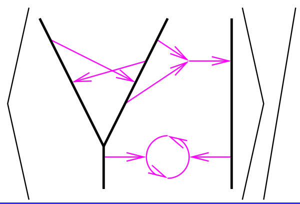
Alekseev



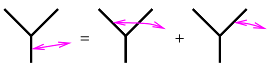
Torossian

Expansions Day 5: Back to 4D — Side 2

w-Jacobi diagrams and $\mathcal{A}$. $\mathcal{A}^w(Y\uparrow) \cong \mathcal{A}^w(\uparrow\uparrow\uparrow)$ is

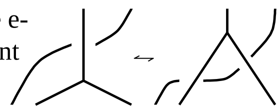


same relations, plus

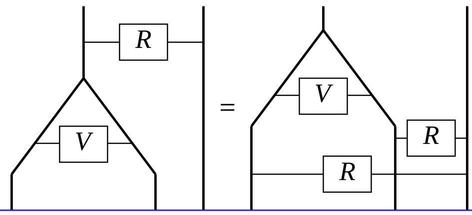
VI: 

deg$=\frac{1}{2}$#{vertices}=6

Knot-Theoretic statement (simplified). There exists a homomorphic expansion Z for trivalent w-tangles. In particular, Z should respect $R4$.



Diagrammatic statement (simplified). Let $R = \exp \overset{\uparrow}{\mapsto} \in \mathcal{A}^w(\uparrow\uparrow)$. There exist $V \in \mathcal{A}^w(\uparrow\uparrow)$ so that:



Algebraic statement (simplified). With $r \in \mathfrak{g}^* \otimes \mathfrak{g}$ the identity element and with $R = e^r \in \hat{\mathcal{U}}(I\mathfrak{g}) \otimes \hat{\mathcal{U}}(\mathfrak{g})$ there exist $V \in \hat{\mathcal{U}}(I\mathfrak{g})^{\otimes 2}$ so that $V(\Delta \otimes 1)(R) = R^{13}R^{23}V$ in $\hat{\mathcal{U}}(I\mathfrak{g})^{\otimes 2} \otimes \hat{\mathcal{U}}(\mathfrak{g})$

Unitary statement (simplified). There exists a unitary tangential differential operator V defined on $\mathrm{Fun}(\mathfrak{g}_x \times \mathfrak{g}_y)$ so that $V\widehat{e^{x+y}} = \widehat{e^x}\widehat{e^y}V$ (allowing $\hat{\mathcal{U}}(\mathfrak{g})$-valued functions)

Group-Algebra statement (simplified). For every $\phi, \psi \in \mathrm{Fun}(\mathfrak{g})^G$ (with small support), the following holds in $\hat{\mathcal{U}}(\mathfrak{g})$:

$$\iint_{\mathfrak{g}\times\mathfrak{g}} \phi(x)\psi(y)e^{x+y} = \iint_{\mathfrak{g}\times\mathfrak{g}} \phi(x)\psi(y)e^x e^y.$$

(shhh, this is Duflo)

Unitary $\implies$ Group-Algebra. $\iint e^{x+y}\phi(x)\psi(y) = \langle 1, e^{x+y}\phi(x)\psi(y)\rangle = \langle V1, Ve^{x+y}\phi(x)\psi(y)\rangle = \langle 1, e^x e^y V\phi(x)\psi(y)\rangle = \langle 1, e^x e^y \phi(x)\psi(y)\rangle = \iint e^x e^y \phi(x)\psi(y)$.

Convolutions statement (Kashiwara-Vergne, simplified). Convolutions of invariant functions on a Lie group agree with convolutions of invariant functions on its Lie algebra. More accurately, let G be a finite dimensional Lie group and let $\mathfrak{g}$ be its Lie algebra, and let $\Phi : \mathrm{Fun}(G) \to \mathrm{Fun}(\mathfrak{g})$ be given by $\Phi(f)(x) := f(\exp x)$. Then if $f, g \in \mathrm{Fun}(G)$ are Ad-invariant and supported near the identity, then $\Phi(f) \star \Phi(g) = \Phi(f \star g)$.

Convolutions and Group Algebras (ignoring all Jacobians). If G is finite, A is an algebra, $\tau : G \to A$ is multiplicative then $(\mathrm{Fun}(G), \star) \to (A, \cdot)$ via $L : f \mapsto \sum f(a)\tau(a)$. For Lie $(G, \mathfrak{g})$,

$$\begin{array}{ccc} (\mathfrak{g},+) \ni x & \xrightarrow{\tau_0=\exp_S} & e^x \in \hat{\mathcal{S}}(\mathfrak{g}) \\ \downarrow \exp_G & \searrow^{\exp_{\mathcal{U}}} & \downarrow \chi \\ (G,\cdot) \ni e^x & \xrightarrow{\tau_1} & e^x \in \hat{\mathcal{U}}(\mathfrak{g}) \end{array} \quad \text{so} \quad \begin{array}{ccc} \mathrm{Fun}(\mathfrak{g}) & \xrightarrow{L_0} & \hat{\mathcal{S}}(\mathfrak{g}) \\ \downarrow \Phi^{-1} & & \downarrow \chi \\ \mathrm{Fun}(G) & \xrightarrow{L_1} & \hat{\mathcal{U}}(\mathfrak{g}) \end{array}$$

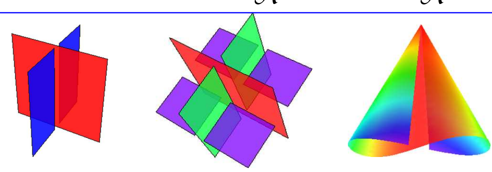
with $L_0\psi = \int \psi(x)e^x dx \in \hat{\mathcal{S}}(\mathfrak{g})$ and $L_1\Phi^{-1}\psi = \int \psi(x)e^x \in \hat{\mathcal{U}}(\mathfrak{g})$. Given $\psi_i \in \mathrm{Fun}(\mathfrak{g})$ compare $\Phi^{-1}(\psi_1) \star \Phi^{-1}(\psi_2)$ and $\Phi^{-1}(\psi_1 \star \psi_2)$ in $\hat{\mathcal{U}}(\mathfrak{g})$:

$$\star \text{ in } G: \iint \psi_1(x)\psi_2(y)e^x e^y \qquad \star \text{ in } \mathfrak{g}: \iint \psi_1(x)\psi_2(y)e^{x+y}$$

$u \leftrightarrow w$ The diagram on the right explains the relationship between associators and solutions of the Kashiwara-Vergne problem.

$$\begin{array}{ccc} KTG & \xrightarrow{a} & wTF \\ \downarrow Z^u & & \downarrow Z^w \\ \mathcal{A}^u & \xrightarrow{\alpha} & \mathcal{A}^w \end{array}$$

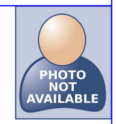
The Full 2-Knot Story



Question. Does it all extend to arbitrary 2-knots (not necessarily "simple")? To arbitrary codimension-2 knots?

BF Following [CR]. $A \in \Omega^1(M = \mathbb{R}^4, \mathfrak{g})$, $B \in \Omega^2(M, \mathfrak{g}^*)$,

$$S(A,B) := \int_M \langle B, F_A\rangle.$$



Rossi

With $\kappa: (S = \mathbb{R}^2) \to M$, $\beta \in \Omega^0(S, \mathfrak{g})$, $\alpha \in \Omega^1(S, \mathfrak{g}^*)$, set

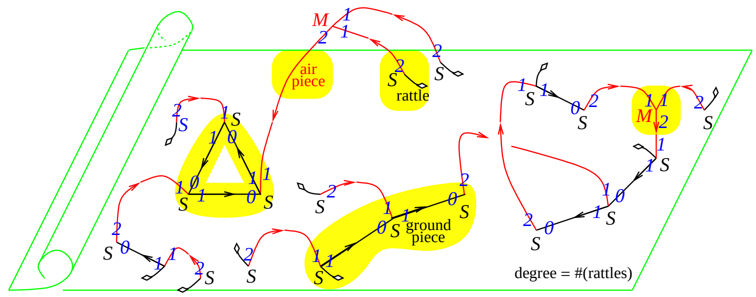
$$\mathcal{O}(A,B,\kappa) := \int \mathcal{D}\beta\mathcal{D}\alpha \exp\left(\frac{i}{\hbar}\int_S \langle\beta, d_{\kappa^*A}\alpha + \kappa^*B\rangle\right).$$

Cattaneo

The BF Feynman Rules. For an edge e, let Φ_e be its direction, in S^3 or S^1. Let ω_3 and ω_1 be volume forms on S^3 and S_1. Then

$$Z_{BF} = \sum_{\substack{\text{diagrams}\\ D}} \frac{[D]}{|\mathrm{Aut}(D)|} \underbrace{\int_{\mathbb{R}^2}\cdots\int_{\mathbb{R}^2}}_{S\text{-vertices}} \underbrace{\int_{\mathbb{R}^4}\cdots\int_{\mathbb{R}^4}}_{M\text{-vertices}} \prod_{\substack{\text{red}\\ e\in D}} \Phi_e^*\omega_3 \prod_{\substack{\text{black}\\ e\in D}} \Phi_e^*\omega_1$$

(modulo some IHX-like relations). See also [Wa]



Issues. • Signs don't quite work out, and BF seems to reproduce only "half" of the wheels invariant on simple 2-knots.
• There are many more configuration space integrals than BF Feynman diagrams and than just trees and wheels.
• I don't know how to define / analyze "finite type" for general 2-knots.
• I don't know how to reduce Z_{BF} to combinatorics / algebra.

References.

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"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)