Commutators

Abstract. The commutator of two elements x and y in a group G is $xyx^{-1}y^{-1}$. That is, x followed by y followed by the inverse of x followed by the inverse of y. In my talk I will tell you how commutators are related to the following four riddles:

1. Can you send a secure message to a person you have never communicated with before (neither privately nor publicly), using a messenger you do not trust?
2. Can you hang a picture on a string on the wall using n nails, so that if you remove any one of them, the picture will fall?
3. Can you draw an n-component link (a knot made of n non-intersecting circles) so that if you remove any one of those n components, the remaining $(n-1)$ will fall apart?
4. Can you solve the quintic in radicals? Is there a formula for the zeros of a degree 5 polynomial in terms of its coefficients, using only the operations on a scientific calculator?

Definition. The commutator of two elements x and y in a group G is $[x,y] := xyx^{-1}y^{-1}$.

Example 1. In S_3, $[(12),(23)] = (12)(23)(12)^{-1}(23)^{-1} = (123)$ and in general in $S_{\geq 3}$,

$$[(ij),(jk)] = (ijk).$$

Example 2. In $S_{\geq 4}$,

$$[(ijk),(jkl)] = (ijk)(jkl)(ijk)^{-1}(jkl)^{-1} = (il)(jk).$$

Example 3. In $S_{\geq 5}$,

$$[(ijk),(klm)] = (ijk)(klm)(ijk)^{-1}(klm)^{-1} = (jkm).$$

Example 4. So, in fact, in S_5, $(123) = [(412),(253)] = [[(341),(152)],[(125),(543)]] = [[[(234),(451)],[(315),(542)]],[[(312),(245)],[(154),(423)]]] = [\,[[[(123),(354)],[(245),(531)]],[[(231),(145)],[(154),(432)]]],[[[(431),(152)],[(124),(435)]],[[(215),(534)],[(142),(253)]]]\,]$.

Solving the Quadratic, $ax^2+bx+c=0$: $\delta = \sqrt{\Delta}$; $\Delta = b^2-4ac$; $r = \frac{\delta-b}{2a}$.

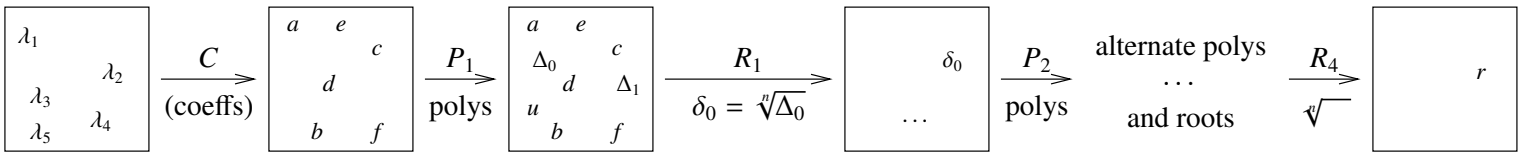
Solving the Cubic, $ax^3+bx^2+cx+d=0$: $\Delta = 27a^2d^2-18abcd+4ac^3+4b^3d-b^2c^2$; $\delta = \sqrt{\Delta}$; $\Gamma = 27a^2d-9abc+3\sqrt{3a\delta}+2b^3$; $\gamma = \sqrt[3]{\frac{\Gamma}{2}}$; $r = -\frac{\frac{b^2-3ac}{\gamma}+b+\gamma}{3a}$.

Solving the Quartic, $ax^4+bx^3+cx^2+dx+e=0$: $\Delta_0 = 12ae-3bd+c^2$; $\Delta_1 = -72ace+27ad^2+27b^2e-9bcd+2c^3$; $\Delta_2 = \frac{1}{27}\left(\Delta_1^2-4\Delta_0^3\right)$; $u = \frac{8ac-3b^2}{8a^2}$; $v = \frac{8a^2d-4abc+b^3}{8a^3}$; $\delta_2 = \sqrt{\Delta_2}$; $Q = \frac{1}{2}\left(3\sqrt{3}\delta_2+\Delta_1\right)$; $q = \sqrt[3]{Q}$; $S = \frac{\frac{\Delta_0}{q}+q}{12a}-\frac{u}{6}$; $s = \sqrt{S}$; $\Gamma = -\frac{v}{s}-4S-2u$; $\gamma = \sqrt{\Gamma}$; $r = -\frac{b}{4a}+\frac{\gamma}{2}+s$.

Theorem. The is no general formula, using only the basic arithmetic operations and taking roots, for the solution of the quintic equation $ax^5+bx^4+cx^3+dx^2+ex+f=0$.

Key Point. The "persistent root" of a closed path (path lift, in topological language) may not be closed, yet the persistent root of a commutators of closed paths is always closed.

Proof. Suppose there was a formula, and consider the corresponding "composition of machines" picture:



$X_0 = \mathbb{C}^5 \setminus \mathcal{D}$ $X_1 = \mathbb{C}^6$ $X_2 = \mathbb{C}^9$ $X_3 = \mathbb{C}^{10}$ $X_9 = \mathbb{C}$

Now if $\gamma_1^{(1)}, \gamma_2^{(1)}, \dots, \gamma_{16}^{(1)}$, are paths in X_0 that induce permutations of the roots and we set $\gamma_1^{(2)} := [\gamma_1^{(1)}, \gamma_2^{(1)}]$, $\gamma_2^{(2)} := [\gamma_3^{(1)}, \gamma_4^{(1)}]$, $\dots$, $\gamma_8^{(2)} := [\gamma_{15}^{(1)}, \gamma_{16}^{(1)}]$, $\gamma_1^{(3)} := [\gamma_1^{(2)}, \gamma_2^{(2)}]$, $\dots$, $\gamma_4^{(3)} := [\gamma_7^{(2)}, \gamma_8^{(2)}]$, $\gamma_1^{(4)} := [\gamma_1^{(3)}, \gamma_2^{(3)}]$, $\gamma_2^{(4)} := [\gamma_3^{(3)}, \gamma_4^{(3)}]$, and finally $\gamma^{(5)} := [\gamma_1^{(4)}, \gamma_2^{(4)}]$ (all of those, commutators of "long paths"; I don't know the word "homotopy"), then $\gamma^{(5)} /\!\!/ C /\!\!/ P_1 /\!\!/ R_1 /\!\!/ \cdots /\!\!/ R_4$ is a closed path. Indeed,

- In X_0, none of the paths is necessarily closed.
- After C, all of the paths are closed.
- After P_1, all of the paths are still closed.
- After R_1, the $\gamma^{(1)}$'s may open up, but the $\gamma^{(2)}$'s remain closed.

...

- At the end, after R_4, $\gamma^{(4)}$'s may open up, but $\gamma^{(5)}$ remains closed.


V.I. Arnold

But if the paths are chosen as in Example 4, $\gamma^{(5)} /\!\!/ C /\!\!/ P_1 /\!\!/ R_1 /\!\!/ \cdots /\!\!/ R_4$ is not a closed path. □

References. V.I. Arnold, 1960s, hard to locate.
V.B. Alekseev, *Abel's Theorem in Problems and Solutions, Based on the Lecture of Professor V.I. Arnold,* Kluwer 2004.
A. Khovanskii, *Topological Galois Theory, Solvability and Unsolvability of Equations in Finite Terms,* Springer 2014.
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The Princess Bride, 1987.

Inigo Montoya: You are using Bonetti's defense against me, uh?

Man in Black: I thought it fitting, considering the rocky terrain.

IM: Naturally, you must expect me to attack with Capo Ferro.

MiB: Naturally, but I find that Thibault cancels out Capo Ferro, don't you?

IM: Unless the enemy has studied his Agrippa, which I have! You are wonderful!

MiB: Thank you. I've worked hard to become so.

IM: I admit it, you are better than I am.

MiB: Then why are you smiling?

IM: Because I know something you don't know.

MiB: And what is that? **IM:** I am not left-handed.

MiB: You're amazing! **IM:** I ought to be after twenty years.

MiB: There is something I ought to tell you. **IM:** Tell me.

MiB: I'm not left-handed either. **IM:** Who are you?

MiB: No one of consequence. **IM:** I must know.

MiB: Get used to disappointment. **IM:** Okay. Kill me quickly.

MiB: I would as soon destroy a stained-glass window as an artist like yourself. However, since I can't have you following me either... Please understand I hold you in the highest respect.

Yes, Prime Minister, 1986.

Sir Humphrey: You know what happens: nice young lady comes up to you. Obviously you want to create a good impression, you don't want to look a fool, do you? So she starts asking you some questions: Mr. Woolley, are you worried about the number of young people without jobs? **Bernard Woolley:** Yes

H: Are you worried about the rise in crime among teenagers? **W:** Yes

H: Do you think there is a lack of discipline in our Comprehensive schools? **W:** Yes

H: Do you think young people welcome some authority and leadership in their lives? **W:** Yes

H: Do you think they respond to a challenge? **W:** Yes

H: Would you be in favour of reintroducing National Service? **W:** Oh...well, I suppose I might be.

H: Yes or no? **W:** Yes

H: Of course you would, Bernard. After all you told me can't say no to that. So they don't mention the first five questions and they publish the last one. **W:** Is that really what they do?

H: Well, not the reputable ones no, but there aren't many of those. So alternatively the young lady can get the opposite result. **W:** How?

H: Mr. Woolley, are you worried about the danger of war? **W:** Yes

H: Are you worried about the growth of armaments? **W:** Yes

H: Do you think there is a danger in giving young people guns and teaching them how to kill? **W:** Yes

H: Do you think it is wrong to force people to take up arms against their will? **W:** Yes

H: Would you oppose the reintroduction of National Service? **W:** Yes

H: There you are, you see Bernard. The perfect balanced sample.