Commutators

Math Union Guest Speaker, January 2015

Abstract. The commutator of two elements $x$ and $y$ in a group $G$ is $xyx^{-1}y^{-1}$. That is, $x$ followed by $y$ followed by the inverse of $x$ followed by the inverse of $y$. In my talk I will tell you how commutators are related to the following four riddles:

1. Can you send a secure message to a person you have never communicated with before (neither privately nor publicly), using a messenger you do not trust?

2. Can you hang a picture on a string on the wall using $n$ nails, so that if you remove any one of them, the picture will fall?

3. Can you draw an $n$-component link (a knot made of $n$ non-intersecting circles) so that if you remove any one of those $n$ components, the remaining $(n-1)$ will fall apart?

4. Can you solve the quintic in radicals? Is there a formula for the zeros of a degree 5 polynomial in terms of its coefficients, using only the operations on a scientific calculator?
Handout
Definitions and Very Simple Examples

**Definition.** The commutator of two elements $x$ and $y$ in a group $G$ is $[x, y] := xyx^{-1}y^{-1}$.

**Example 1.** In $S_3$, $[(12), (23)] = (12)(23)(12)^{-1}(23)^{-1} = (123)$ and in general in $S_{n}$, $[(ij), (jk)] = (ijk)$.

**Example 2.** In $S_4$, $[(ijk), (jkl)] = (ijk)(jkl)(ijk)^{-1}(jkl)^{-1} = (il)(jk)$.

**Example 3.** In $S_5$, $[(ijk), (klm)] = (ijk)(klm)(ijk)^{-1}(klm)^{-1} = (jkm)$. 
Problem #1

Can you send a secure message to a person you have never communicated with before (neither privately nor publicly), using a messenger you do not trust?

(Image from http://cs.wellesley.edu/~cs110/OLD_WEBSITE/lectures/L18-encryption/handout.html)
Problem #2

Can you hang a picture on a string on the wall using \( n \) nails, so that if you remove any one of them, the picture will fall?
Problem #3

Can you draw an \( n \)-component link (a knot made of \( n \) non-intersecting circles) so that if you remove any one of those \( n \) components, the remaining \((n-1)\) will fall apart?

\[
\text{Module}[\{n = 120, \ a = 2, \ w = 0.3\},
\begin{align*}
\text{Graphics3D}\{ & \text{Red, Tube[Table[\{a Cos[t], Sin[t], 0\}, \{t, 0, 2 \pi, 2 \pi/n\}], w]}, \\
& \text{Green, Tube[Table[\{0, a Cos[t], Sin[t]\}, \{t, 0, 2 \pi, 2 \pi/n\}], w]}, \\
& \text{Blue, Tube[Table[\{Sin[t], 0, a Cos[t]\}, \{t, 0, 2 \pi, 2 \pi/n\}], w]}
\}
\}, \text{Boxed \rightarrow False}\]
\]
Problem #4 - Our Main Topic

\[ ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0 \]

Can you solve the quintic in radicals? Is there a formula for the zeros of a degree 5 polynomial in terms of its coefficients, using only the operations on a scientific calculator? \((+, -, \times, \div, \sqrt{a})\)

History: First solved by Abel / Galois in the 1800s. Our solution follows Arnold's topological solution from the 1960s. I could not find the original writeup by Arnold (if it at all exists), yet see:


A Sword Fight


Inigo: You are using Bonetti’s defense against me, uh?
Man In Black: I thought it fitting, considering the rocky terrain.
Inigo: Naturally, you must expect me to attack with Capo Ferro.
Man In Black: Naturally, but I find that Thibault cancels out Capo Ferro, don’t you?
Inigo: Unless the enemy has studied his Agrippa, which I have! You are wonderful!
Man In Black: Thank you. I’ve worked hard to become so.
Inigo: I admit it, you are better than I am.
Man In Black: Then why are you smiling?
Inigo: Because I know something you don’t know.
Man In Black: And what is that?
Inigo: I am not left-handed.
Man In Black: You’re amazing!
Inigo: I ought to be after twenty years.
Man In Black: There is something I ought to tell you.
Inigo: Tell me.
Man In Black: I’m not left-handed either.
Inigo: Who are you?
Man In Black: No one of consequence.
Inigo: I must know.
Man In Black: Get used to disappointment.
Inigo: Okay.
Inigo: Kill me quickly.
Man In Black: I would as soon destroy a stained-glass window as an artist like yourself. However, since I can’t have you following me either....
Man In Black: Please understand I hold you in the highest respect.
Solving the Quadratic $ax^2 + bx + c = 0$

$$\Delta = b^2 - 4ac;$$

$$\delta = \sqrt{\Delta};$$

$$r = (-b + \delta) \div (2a)$$
Testing the Quadratic Solution
Square Roots and Persistent Square Roots
Leading Questions

“Yes, Prime Minister”, 1986.

Sir Humphrey: You know what happens: nice young lady comes up to you. Obviously you want to create a good impression, you don’t want to look a fool, do you? So she starts asking you some questions: Mr. Woolley, are you worried about the number of young people without jobs?
Bernard Woolley: Yes
Sir Humphrey: Are you worried about the rise in crime among teenagers?
Bernard Woolley: Yes
Sir Humphrey: Do you think there is a lack of discipline in our Comprehensive schools?
Bernard Woolley: Yes
Sir Humphrey: Do you think young people welcome some authority and leadership in their lives?
Bernard Woolley: Yes
Sir Humphrey: Do you think they respond to a challenge?
Bernard Woolley: Yes
Sir Humphrey: Would you be in favour of reintroducing National Service?
Bernard Woolley: Oh...well, I suppose I might be.
Sir Humphrey: Yes or no?
Bernard Woolley: Yes
Sir Humphrey: Of course you would, Bernard. After all you told me can’t say no to that. So they don’t mention the first five questions and they publish the last one.
Bernard Woolley: Is that really what they do?
Sir Humphrey: Well, not the reputable ones no, but there aren’t many of those. So alternatively the young lady can get the opposite result.
Bernard Woolley: How?
Sir Humphrey: Mr. Woolley, are you worried about the danger of war?
Bernard Woolley: Yes
Sir Humphrey: Are you worried about the growth of armaments?
Bernard Woolley: Yes
Sir Humphrey: Do you think there is a danger in giving young people guns and teaching them how to kill?
Bernard Woolley: Yes
Sir Humphrey: Do you think it is wrong to force people to take up arms against their will?
Bernard Woolley: Yes
Sir Humphrey: Would you oppose the reintroduction of National Service?
Bernard Woolley: Yes
Sir Humphrey: There you are, you see Bernard. The perfect balanced sample.
Solving the Cubic $ax^3 + bx^2 + cx + d = 0$

$$
\Delta = -18\ a\ b\ c\ d + 4\ b^3\ d - b^2\ c^2 + 4\ a\ c^3 + 27\ a^2\ d^2;
$$
$$
\delta = \sqrt{\Delta} ;
$$
$$
\Gamma = 2\ b^3 - 9\ a\ b\ c + 27\ a^2\ d + 3\ \sqrt{3}\ a\ \delta ;
$$
$$
\gamma = \sqrt[3]{\Gamma / 2} ;
$$
$$
r = - (b + \gamma + (b^2 - 3\ a\ c) / \gamma) / (3\ a) 
$$
Testing the Cubic Solution
Solving the Quartic $ax^4 + bx^3 + cx^2 + dx + e = 0$

\[ \Delta_0 = c^2 - 3b d + 12 a e; \]
\[ \Delta_1 = 2 c^3 - 9 b c d + 27 b^2 e + 27 a d^2 - 72 a c e; \]
\[ \Delta_2 = (-4 \Delta_0^3 + \Delta_1^2) / 27; \]
\[ u = (8 a c - 3 b^2) / (8 a^2); \]
\[ v = (b^3 - 4 a b c + 8 a^2 d) / (8 a^3); \]
\[ \delta_2 = \sqrt[3]{\Delta_2}; \]
\[ Q = \left( \Delta_1 + 3 \sqrt[3]{\delta_2} \right) / 2; \]
\[ q = \sqrt{Q}; \]
\[ S = -u / 6 + (q + \Delta_0 / q) / (12 a); \]
\[ s = \sqrt{S}; \]
\[ \Gamma = -4 S - 2 u - v / s; \]
\[ \gamma = \sqrt{\Gamma}; \]
\[ r = -b / (4 a) + s + \gamma / 2 \]
Testing the Quartic Solution
Theorem

No such machine exists for the quintic,

\[ ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0. \]
The 10th Root

The Key Point

If a closed path is the commutator of two closed paths, its persistent root is a closed path.
My Name is Inigo Montoya

"The Princess Bride", 1987:

Count Rugen: Good heavens. Are you still trying to win? You've got an overdeveloped sense of vengeance. It's going to get you into trouble someday.


Count Rugen: Stop saying that!

Inigo: HELLO. MY NAME IS INIGO MONTOYA. YOU KILLED MY FATHER, PREPARE TO DIE.

Count Rugen: No!

Inigo: Offer me money!

Count Rugen: Yes!

Inigo: Power, too. Promise me that!

Count Rugen: All that I have and more! Please!

Inigo: Offer me everything I ask for!

Count Rugen: Anything you want.

Inigo: I want my father back, you son of a bitch.