Abstract. I will describe a computable, non-commutative invariant of tangles with values in wheels, almost generalize it to some balloons, and then tell you why I care. Spoilers: tangles are what you know what, wheels are linear combinations of cyclic words in some alphabet, balloons are 2-knots, and one reason I care is because quantum field theory predicts more than I can actually get (but also less).

Why I like "non-commutative"? With $F\mathcal{L}(x)$ the free associative non-commutative algebra, 
$$\dim \mathbb{Q}[x,y]_d \approx 2^d \sim \dim F\mathcal{L}(x,y)_d.$$

Why I like "computable"?
• Because I’m weird.
• Note that $\pi_1$ isn’t computable.

Preliminaries from Algebra. $F\mathcal{L}(x)$ denotes the free Lie algebra in $(x_i)$; $F\mathcal{L}(x) = \langle \text{binary trees with AS vertices and coloured leaves}/(\text{IHX relations}) \rangle$. There an obvious map $F\mathcal{L}(F\mathcal{L}(x)) \to F\mathcal{L}(x)$ defined by $[a,b] \to ab - ba$, which in itself is IHX.

$CW(x)$ denotes the vector space of cyclic words in $(x_i)$: $CW(x) = \mathbb{F}((x_i)/(x_i w = w x_i))$. There an obvious map $CW(F\mathcal{L}(x)) \to CW(x)$. In fact, connected uni-trivalent 2-in-1-out graphs with univalents in colours in $[1, \ldots, n]$, modulo AS and IHX, is precisely $CW(x)$.

Most important. $e^f = \sum \frac{x^d}{d!}$ and $e^{x+y} = e^xe^y$.

Preliminaries from Knot Theory.

Theorem. $\omega$, the connected part of the procedure below, is an invariant of $S$-component tangles with values in $CW(S)$:

$$\sum_{\lambda,\mu} \left( \left\langle \lambda \left| \omega \right| \mu \right\rangle \right)^2 \frac{1}{\text{KLM}!}.$$

Further Facts

• $\omega$ is really the second part of a (trees,wheels)-valued invariant $\zeta = (\lambda, \omega)$. The tree part $\lambda$ is just a repacking of the Milnor $\mu$-invariants.
• On $u$-tangles, $\zeta$ is equivalent to the trees&wheels part of the Kontsevich integral, except it is computable and is defined with no need for a choice of parenthesization.
• On long/round $u$-knots, $\omega$ is equivalent to the Alexander polynomial.
• The multivariable Alexander polynomial (and Levine’s factorization thereof [Le]) is contained in the Abelianization of $\zeta$ [BNS].
• $\omega$ vanishes on braids.
• Related to $f$ extends Farber’s [Fa]?
• Should be summed and categorified.
• Extends to $v$ and descends to $w$: meaning, $\zeta$ satisfies $\omega$ also satisfies $\omega$’s “true domain” is

A: \[
\begin{align*}
\mathcal{A} & \quad e^{\mathcal{A} \mathcal{A}} = e^{\mathcal{A} \mathcal{A}} \\
\mathcal{A} & \quad e^{-\mathcal{A} \mathcal{A}} = e^{-\mathcal{A} \mathcal{A}}
\end{align*}
\]
**Tangles, Wheels, Balloons — 2**

**Question.** Does it all extend to arbitrary 2-knots (not necessarily “simple”)? To arbitrary codimension-2-knots?

**BF Following** [CR]. $A \in \Omega^2(M = \mathbb{R}^4, \gamma), B \in \Omega^2(M, \eta)$, 

$$S(A, B) := \int_M (B, F_A).$$

With $\kappa: (S = \mathbb{R}^3) \to M, \beta \in \Omega^2(S, \eta)$, $\alpha \in \Omega^1(S, \eta)$, set 

$$\mathcal{O}(A, B, \kappa) := \int_{\mathbb{R}^3} \beta \wedge d\alpha \exp\left( i \frac{1}{h} \int_{\mathbb{R}^3} (\beta, d\alpha + \kappa' B).\right)$$

**The BF Feynman Rules.** For an edge $e$, let $\Phi_e$ be its direction, in $S^3$ or $S^1$. Let $\omega_e$ and $\lambda_e$ be volume forms on $S^3$ and $S^1$. Then 

$$Z_{BF} = \sum_{e \in [D]} \left( \frac{[D]}{|\text{Aut}(D)|} \right) \int_{S^3} \cdots \int_{S^3} \int_{S^1} \cdots \int_{S^1} \prod_{e \in \partial V} \Phi_{\omega_e} \prod_{e' \in \partial V} \Phi_{\lambda_e}$$

(nuodulo some IHX-like relations).

**See also [Wa]**

**Issues.** • Signs don’t quite work out, and BF seems to reproduce only “half” of the wheels invariant on simple 2-knots.

• There are many more configuration space integrals than BF Feynman diagrams and just trees and wheels.

• I don’t know how to define / analyze “finite type” for general knots.

• I don’t know how to reduce $Z_{BF}$ to combinatorics / algebra.

**References.**


“God created the knots, all else in topology is the work of mortals.”

Leopold Kronecker (modified)