Tangles, Wheels, Balloons

Abstract. I will describe a computable, non-commutative invariant of tangles with values in wheels, almost generalize it to some balloons, and then tell you why I care. Spoilers: tangles are you know what, wheels are linear combinations of cyclic words in some alphabet, balloons are 2-knots, and one reason I care is because quantum field theory predicts more than I can actually get (but also less).

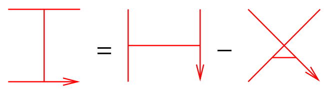
Why I like "non-commutative"? With $FA(x_i)$ the free associative non-commutative algebra,

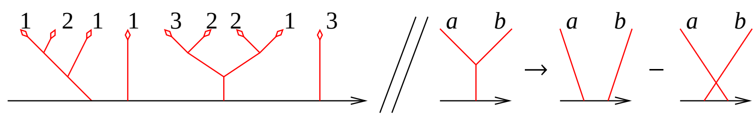
$$\dim \mathbb{Q}[x,y]_d \sim d \ll 2^d \sim \dim FA(x,y)_d.$$

Why I like "computable"?

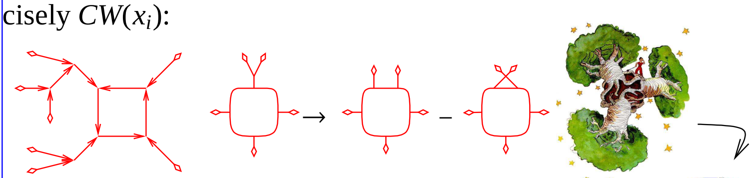
- Because I'm weird.
- Note that π_1 isn't computable.

Preliminaries from Algebra. $FL(x_i)$ denotes the free Lie algebra in (x_i); $FL(x_i)$ = (binary trees with AS vertices and coloured leafs)/(IHX relations). There an obvious map $FA(FL(x_i)) \to FA(x_i)$ defined by $[a,b] \to ab - ba$, which in itself, is IHX.

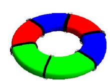




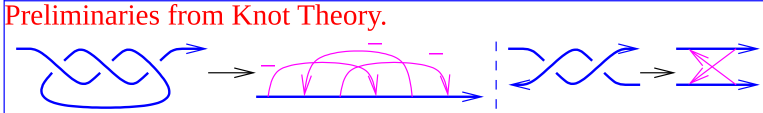
$CW(x_i)$ denotes the vector space of cyclic words in (x_i): $CW(x_i) = FA(x_i)/(x_i w = w x_i)$. There an obvious map $CW(FL(x_i)) \to CW(x_i)$. In fact, connected uni-trivalent 2-in-1-out graphs with univalents with colours in $\{1,\dots,n\}$, modulo AS and IHX, is precisely $CW(x_i)$:



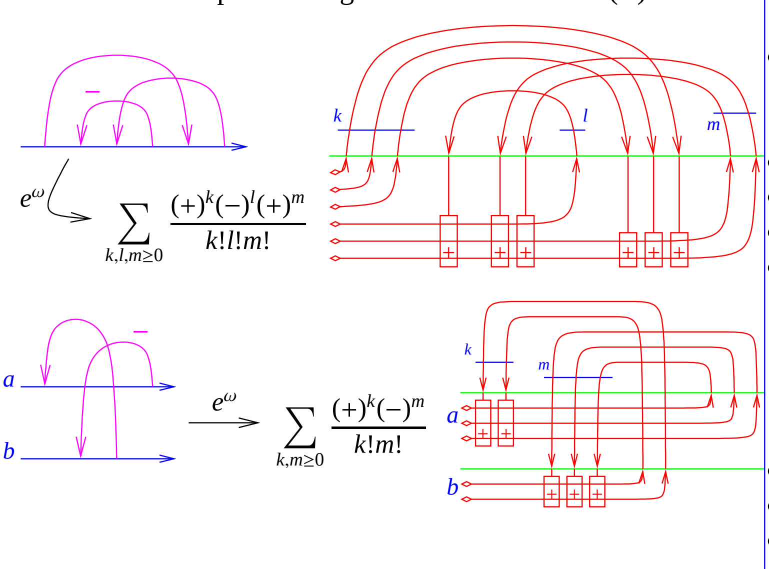
Most important. $e^x = \sum \frac{x^d}{d!}$ and $e^{x+y} = e^x e^y$.

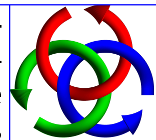


Preliminaries from Knot Theory.

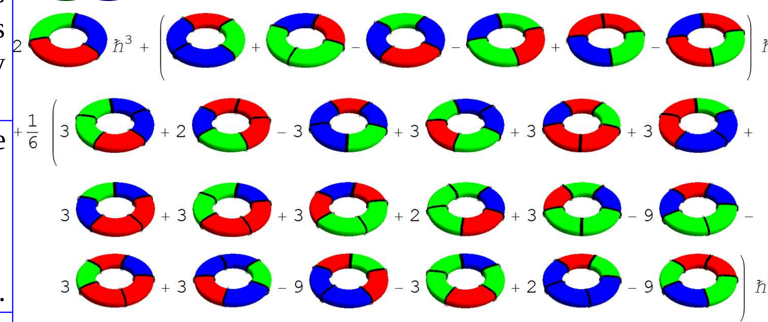


Theorem. ω, the connected part of the procedure below, is an invariant of S-component tangles with values in $CW(S)$:



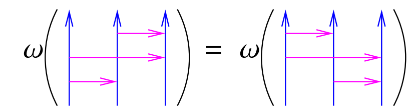


ω is practically computable! For the Borromean tangle, to degree 5, the result is: (see [BN])

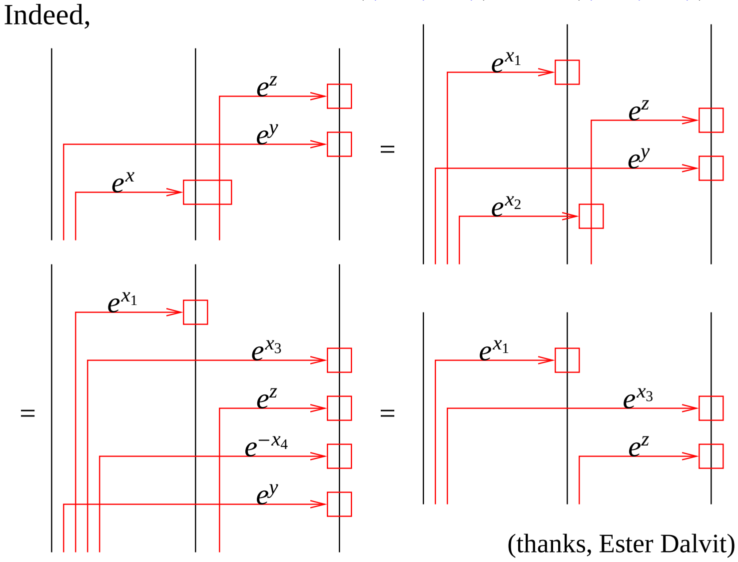


Proof of Invariance.

Need to show:



Indeed,



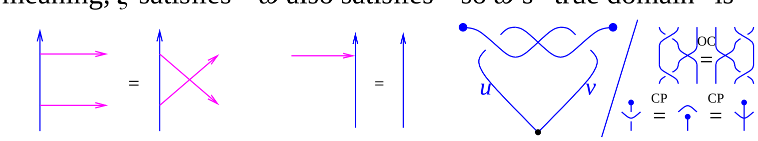
(thanks, Ester Dalvit)

Further Facts

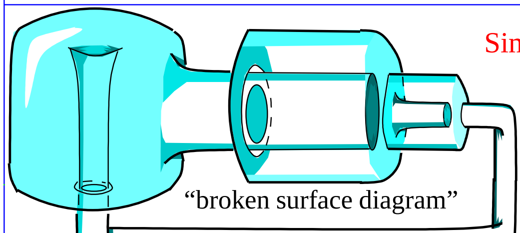
- ω is really the second part of a (trees,wheels)-valued invariant $\zeta = (\lambda, \omega)$. The tree part λ is just a repackaging of the Milnor μ-invariants.
- On u-tangles, ζ is equivalent to the trees&wheels part of the Kontsevich integral, except it is computable and is defined with no need for a choice of parenthesization.
- On long/round u-knots, ω is equivalent to the Alexander polynomial.
- The multivariable Alexander polynomial (and Levine's factorization thereof [Le]) is contained in the Abelianization of ζ [BNS].
- ω vanishes on braids.
- Related to / extends Farber's [Fa]?
- Should be summed and categorified.
- Extends to v and descends to w:

Does ω extend to balloons?

meaning, ζ satisfies ω also satisfies   so ω's "true domain" is

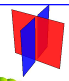


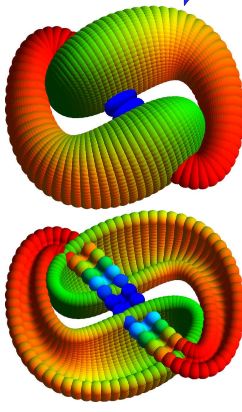
- Agrees with BN-Dancso [BND1, BND2] and with [BN].
- ζ, ω are universal finite type invariants.
- Using ⋊K: $v\mathcal{K}_n \to w\mathcal{K}_{n+1}$, defines a strong invariant of v-tangles / long v-knots. (⋊K in LaTeX: ωεβ/zhe)



A 4D knot by Carter and Saito [CS]

Simple 2-Knots.



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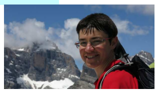
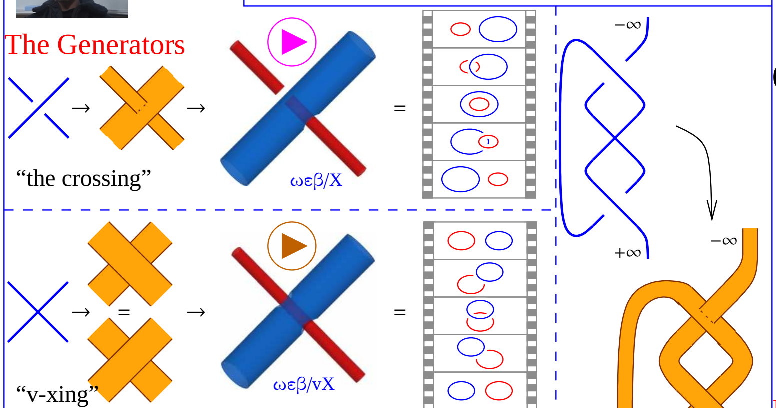
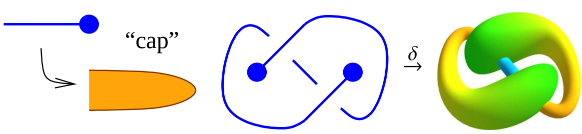


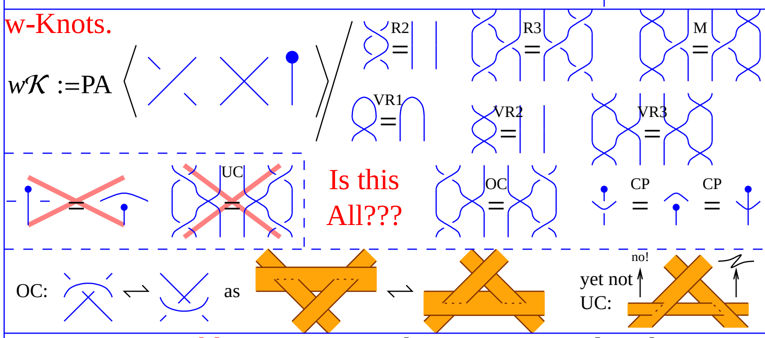
The Generators





The Double Inflation Procedure δ.

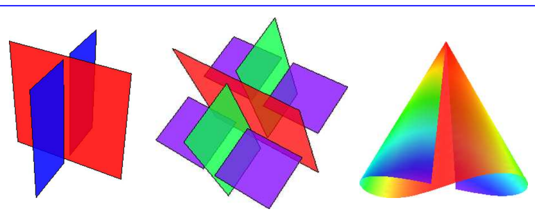
w-Knots.



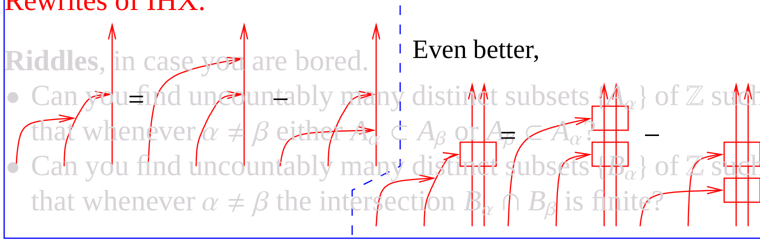
A Big Open Problem. δ maps w-knots onto simple 2-knots. To what extent is it a bijection? What other relations are required? In other words, **find a simple description of simple 2-knots**.

The Full

2-Knot Story



Rewrites of IHX.



Even better,

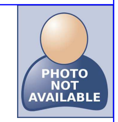
Riddles, in case you are bored.

- Can you find uncountably many distinct subsets $\{A_\alpha\}$ of $\mathbb{Z}$ such that whenever $\alpha \neq \beta$ either $A_\alpha \subset A_\beta$ or $A_\beta \subset A_\alpha$?
- Can you find uncountably many distinct subsets $\{B_\alpha\}$ of $\mathbb{Z}$ such that whenever $\alpha \neq \beta$ the intersection $B_\alpha \cap B_\beta$ is finite?

Question. Does it all extend to arbitrary 2-knots (not necessarily "simple")? To arbitrary codimension-2 knots?

BF Following [CR]. $A \in \Omega^1(M = \mathbb{R}^4, \mathfrak{g})$, $B \in \Omega^2(M, \mathfrak{g}^*)$,

$$S(A,B) := \int_M \langle B, F_A\rangle.$$


Rossi

With $\kappa\colon (S = \mathbb{R}^2) \to M$, $\beta \in \Omega^0(S,\mathfrak{g})$, $\alpha \in \Omega^1(S,\mathfrak{g}^*)$, set

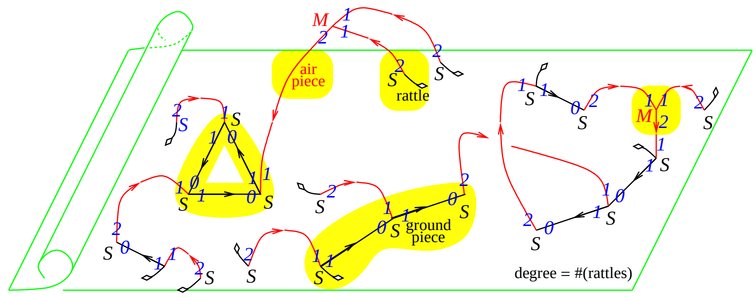
$$\mathcal{O}(A,B,\kappa) := \int \mathcal{D}\beta\mathcal{D}\alpha \exp\left(\frac{i}{\hbar}\int_S \langle \beta, d_{\kappa^* A}\alpha + \kappa^* B\rangle\right).$$


Cattaneo

The BF Feynman Rules. For an edge e, let Φ_e be its direction, in S^3 or S^1. Let ω_3 and ω_1 be volume forms on S^3 and S_1. Then

$$Z_{BF} = \sum_{\substack{\text{diagrams}\\ D}} \frac{[D]}{|\mathrm{Aut}(D)|} \underbrace{\int_{\mathbb{R}^2}\cdots\int_{\mathbb{R}^2}}_{S\text{-vertices}} \underbrace{\int_{\mathbb{R}^4}\cdots\int_{\mathbb{R}^4}}_{M\text{-vertices}} \prod_{\substack{\text{red}\\ e\in D}} \Phi_e^*\omega_3 \prod_{\substack{\text{black}\\ e\in D}} \Phi_e^*\omega_1$$

(modulo some IHX-like relations). See also [Wa]



Issues. • Signs don't quite work out, and BF seems to reproduce only "half" of the wheels invariant on simple 2-knots.
• There are many more configuration space integrals than BF Feynman diagrams and than just trees and wheels.
• I don't know how to define / analyze "finite type" for general 2-knots.
• I don't know how to reduce Z_{BF} to combinatorics / algebra.

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"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)