Abstract. I will describe a computable, non-commutative invariant of tangles with values in wheels, almost generalize it to some balloons, and then tell you why I care. Spoilers: tangles are you know what, wheels are linear combinations of cyclic words in some alphabet, balloons are 2-knots, and one reason I care is because quantum field theory predicts more than I can actually get (but also less).

Why I like “non-commutative”? With \( FA(x) \) the free associative non-commutative algebra,

\[
\dim \mathbb{Q}[x, y]_d \sim d \ll 2^d \sim \dim FA(x, y)_{ij}.
\]

Why I like “computable”? 
- Because I’m weird.
- Note that \( \pi_1 \) isn’t computable.

Preliminaries from Algebra. \( FL(x) \) denotes the free Lie algebra in \( (x) \);

\[
FL(x) = (\text{binary trees with AS vertices and coloured leafs})/(\text{IHX relations}).
\]

There an obvious map \( FA(FL(x)) \to FA(x) \) defined by \( [a, b] \to ab - ba \), which in itself, is IHX.

\[
CW(x) \text{ denotes the vector space of cyclic words in } (x): \quad CW(x) = FA(x)/\langle x, w = wx \rangle.
\]

There an obvious map \( CW(FL(x)) \to CW(x) \). In fact, connected uni-trivalent 2-in-1-out graphs with univalents with colours in \( \{1, \ldots, n\} \), modulo AS and IHX, is precisely \( CW(x) \):

Most important. \( e^x = \sum \frac{x^d}{d!} \) and \( e^{x+y} = e^xe^y \).

Preliminaries from Knot Theory.

Theorem. \( \omega \), the connected part of the procedure below, is an invariant of \( S \)-component tangles with values in \( CW(S) \):

\[
\omega \approx \sum_{k, \lambda \geq 0} \frac{(+1)^z}{k!l!m!} \cdot 4\pi
\]

Indeed,

\[
\omega \text{ is really the second part of a (trees,wheels)-valued \text{ invariant } } \zeta = (\lambda, \omega). \text{ The tree part } \lambda \text{ is just a repacking of the Milnor } \mu \text{-invariants.}
\]

Further Facts
- On u-tangles, \( \zeta \) is equivalent to the trees&wheels part of the Kontsevich integral, except it is computable and is defined with no need for a choice of parenthesization.
- On long/round u-knots, \( \omega \) is equivalent to the Alexander polynomial.
- The multivariable Alexander polynomial (and Levine’s factorization thereof \( [Le] \)) is contained in the Abelianization of \( \zeta \) \( [BNS] \).
- \( \omega \) vanishes on braids.
- Related to \( k \) extends Farber’s \( [Fa] \)?
- Should be summed and categorified.
- Extends to \( v \) and descends to \( w \):

\[
\begin{align*}
\omega \text{ also satisfies } & \quad \text{meaning, } \zeta \text{ satisfies } \\
\text{Agrees with BN-Dancso } [BND1, BND2] \text{ and with } [BN].
\end{align*}
\]

Using \( \mathcal{K}: vK_n \to wK_{n+1} \), defines a strong invariant of \( v \)-tangles / long \( v \)-knots.
A Big Question. Does it all extend to arbitrary 2-knots (not necessarily "simple")? To arbitrary codimension-2 knots?

BF Following [CR]. $A \in \Omega^1(M = \mathbb{R}^4, g)$, $B \in \Omega^2(M, g^*)$.

$$S(A, B) := \int_M \langle B, F_A \rangle.$$ 

With $\kappa: (S = \mathbb{R}^2) \to M$, $\beta \in \Omega^0(S, g)$, $\alpha \in \Omega^1(S, g^*)$, set

$$O(A, B, \kappa) := \int D\beta D\alpha D\kappa \exp \left( \frac{i}{\hbar} \int_S \langle B, d e_A \alpha + \kappa^* B \rangle \right).$$

The BF Feynman Rules. For an edge $e$, let $\Phi_e$ be its direction, in $S^3$ or $S^1$. Let $\omega_3$ and $\omega_1$ be volume forms on $S^3$ and $S^1$. Then

$$Z_{BF} = \sum_{\text{diagrams } D} \frac{|D|}{|\text{Aut}(D)|} \int_{S^3} \cdots \int_{S^3} \int_{S^1} \cdots \int_{S^1} \prod_{e \in D} \Phi_e^{\omega_3} \prod_{e \in D} \Phi_e^{\omega_1}$$

(modulo some IHX-like relations).

Issues. • Signs don’t quite work out, and BF seems to reproduce only “half” of the wheels invariant.

• There are many more configuration space integrals than BF Feynman diagrams and than just trees and wheels.

• I don’t know how to define “finite type” for arbitrary 2-knots.

References.


“All created the knots, all else in topology is the work of mortals.”

Leopold Kronecker (modified)

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