Tangles, Wheels, Balloons

Abstract. I will describe a computable, non-commutative invariant of tangles with values in wheels, almost generalize it to some balloons, and then tell you why I care. Spoilers: tangles are you know what, wheels are linear combinations of cyclic words in some alphabet, balloons are 2-knots, and one reason I care is because quantum field theory predicts more than I can actually get (but also less).

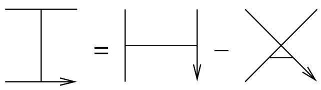
Why I like "non-commutative"? With $FA(x_i)$ the free associative non-commutative algebra,

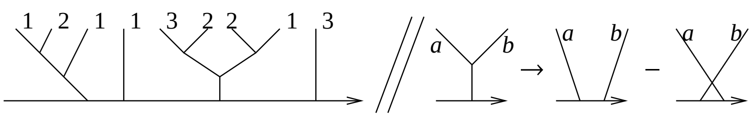
$$\dim \mathbb{Q}[x,y]_d \sim d \ll 2^d \sim \dim FA(x,y)_d.$$

Why I like "computable"?

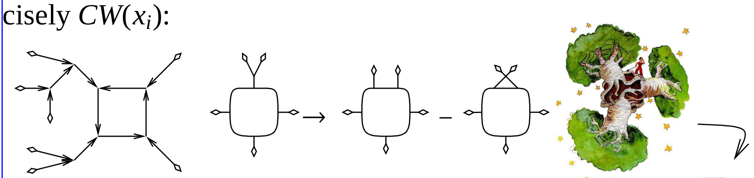
- Because I'm weird.
- Note that π_1 isn't computable.

Preliminaries from Algebra. $FL(x_i)$ denotes the free Lie algebra in (x_i); $FL(x_i)$ = (binary trees with AS vertices and coloured leafs)/(IHX relations). There an obvious map $FA(FL(x_i)) \to FA(x_i)$ defined by $[a,b] \to ab - ba$, which in itself, is IHX.

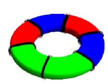




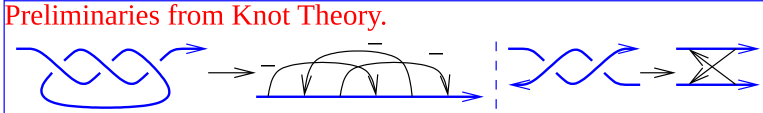
$CW(x_i)$ denotes the vector space of cyclic words in (x_i): $CW(x_i) = FA(x_i)/(x_i w = w x_i)$. There an obvious map $CW(FL(x_i)) \to CW(x_i)$. In fact, connected uni-trivalent 2-in-1-out graphs with univalents with colours in $\{1,\ldots,n\}$, modulo AS and IHX, is precisely $CW(x_i)$:



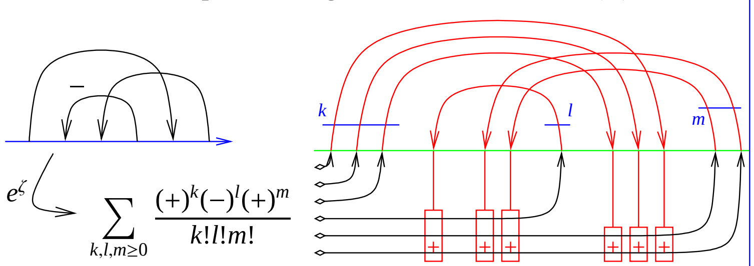
Most important. $e^x = \sum \frac{x^d}{d!}$ and $e^{x+y} = e^x e^y$.

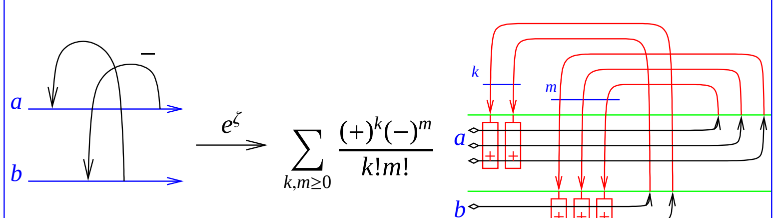


Preliminaries from Knot Theory.

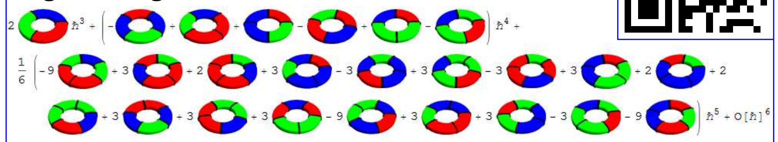


Theorem. ζ, the connected part of the procedure below, is an invariant of S-component tangles with values in $CW(S)$:

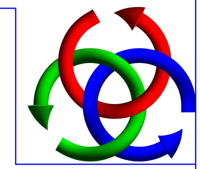


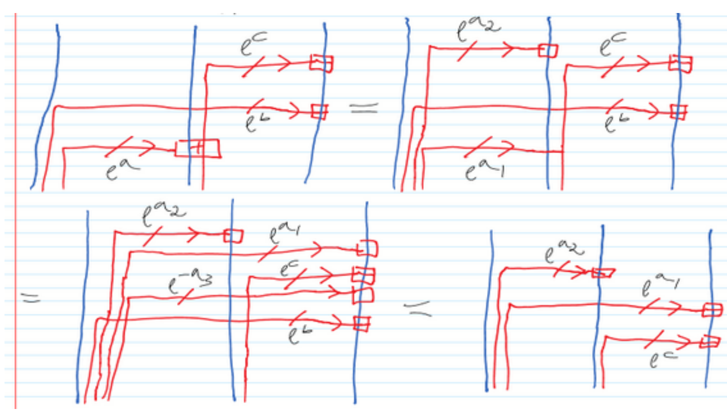


ζ is practically computable! For the Borromean tangle, to degree 5, the result is:



Proof of Invariance.





• Agrees with BN-Dancso [BND1, BND2] and with [BN]. • In-practice computable! • Vanishes on braids. • Extends to w. • Contains Alexander. • The "missing factor" in Levine's factorization [Le] (the rest of [Le] also fits, hence contains the MVA). • Related to / extends Farber's [Fa]? • Should be summed and categorified.

References.

[BN] D. Bar-Natan, *Balloons and Hoops and their Universal Finite Type Invariant, BF Theory, and an Ultimate Alexander Invariant,* http://www.math.toronto.edu/~drorbn/papers/KBH/, arXiv:1308.1721.

[BND1] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects I: W-Knots and the Alexander Polynomial,* http://drorbn.net/AcademicPensieve/Projects/WKO1, arXiv:1405.1956.

[BND2] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects II: Tangles and the Kashiwara-Vergne Problem,* http://drorbn.net/AcademicPensieve/Projects/WKO2, arXiv:1405.1955.

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[Le] J. Levine, *A Factorization of the Conway Polynomial,* Comment. Math. Helv. **74** (1999) 27–53, arXiv:q-alg/9711007.

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"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

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