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Dror Bar-Natan: Talks: Fields-1411: **Finite Type Invariants of Doodles, I**

Abstract. I will describe my former student's Jonathan Zung work on finite type invariants of "doodles", plane curves modulo the second Reidemeister move but not modulo the third. We use a definition of "finite type" different from Arnold's and more along the lines of Goussarov's "Interdependent Modifications", and come to a conjectural combinatorial description of the set of all such invariants. We then describe how to construct many such invariants (though perhaps not all) using a certain class of 2-dimensional "configuration space integrals".

An unfinished project!




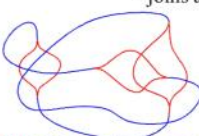
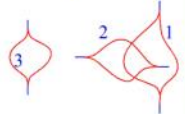
Doodles.

$\mathcal{K} = \mathcal{K}_0 = \mathbb{Q} \langle \text{doodles} \rangle / \langle R2 \rangle$ yet not $R1/R3$

Easy to classify!

Chord Diagrams and an Upper Bound on $\mathcal{K}_n/\mathcal{K}_{n+1}$

The Rayman Principle. In $\mathcal{K}_n/\mathcal{K}_{n+1}$, "joins are irrelevant"

Rings can be subdivided until each one participates in just one "feature".

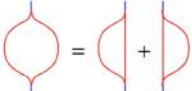
Prior Art. Arnold [Ar] first studied doodles within his study of plane curves and the "strangeness" St invariant. Vassiliev [Va1, Va2] defined finite type invariants in a different way, and Merkov [Me] proved that they separate doodles.

Goussarov Finite-Type.

$\mathcal{K}_n = \langle \text{doodles} \rangle / \langle \text{ring}, \text{join} \rangle$

doodles and detours (dnd's)

The Subdivision Relations. In $\mathcal{K}_n/\mathcal{K}_{n+1}$,



AS: Tetrahedron (Tet) Ring Exchange (RE)

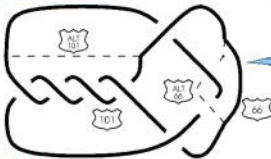
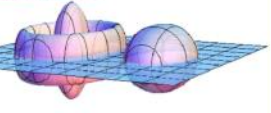
AS: $\sum_{\text{cyclic perms}} \pm \text{diagram} = 0$

Tet: $\sum_{\text{cyclic perms}} \pm \text{diagram} = 0$

RE: $\text{diagram} = \text{diagram}$

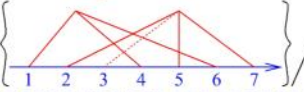
Def. V is of type n if it vanishes on \mathcal{K}_{n+1} . $(\mathcal{K}_0/\mathcal{K}_{n+1})^* \leftrightarrow \mathcal{K}_n/\mathcal{K}_{n+1}$

Knots in 3D. **2-Knots in 4D.**

The reason I care!

"Chord Diagrams".

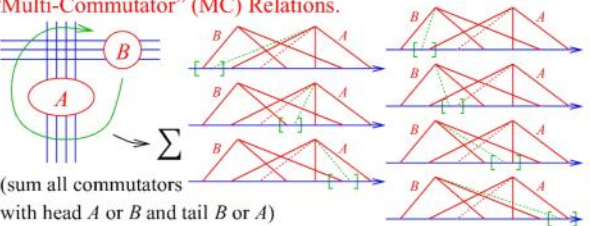


AS Tet RE $\rightarrow \mathcal{K}_n/\mathcal{K}_{n+1}$

Goals.

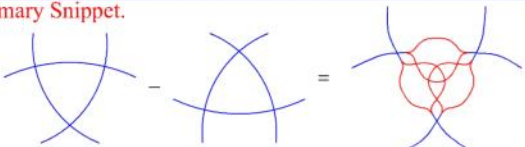
- Describe $\mathcal{A}_n := \mathcal{K}_n/\mathcal{K}_{n+1}$ using diagrams/relations.
- Get many or all finite type invariants of doodles using configuration space integrals.
- Do these come from a TQFT?
- See if \mathcal{A}_n has a "Lie theoretic" (tensors/relations) meaning.
- See if/how Arnold's St and the Merkov invariants integrate in.

"Multi-Commutator" (MC) Relations.



(sum all commutators with head A or B and tail B or A)

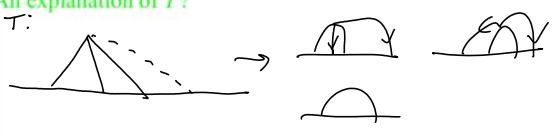
The Primary Snippet.



Summary Diagram.

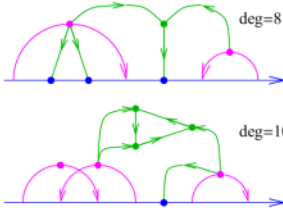
$$\begin{array}{ccc} \mathcal{K} & & \\ \downarrow \cong & \searrow Z & \\ \mathcal{D}^c/MC =: \mathcal{A}^c & \xrightarrow{\pi} \text{gr} \mathcal{K} & \xrightarrow{\text{gr} Z} \mathcal{A}' := \mathcal{D}'/FDR \\ & \searrow T: 1-1? & \end{array}$$

An explanation of T ?

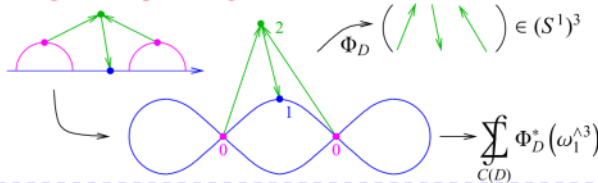


Feynman Diagrams and a Lower Bound on $(\mathcal{K}_0/\mathcal{K}_{n+1})^*$.

Feynman Diagrams. A blue "skeleton line" at the bottom. A magenta "arrow diagram" (directed pairing of skeleton points) on top, with a magenta dot at the middle of each arrow. A green directed graph on top, with 2-in 1-out antisymmetric green vertices, with arbitrary number of green edges starting at the magenta dots, and with some green edges terminating at distinct blue skeleton points. The degree is the total valency of the magenta dots.



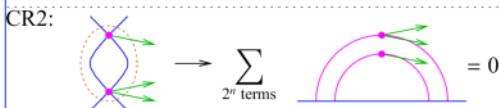
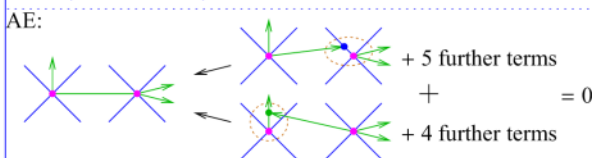
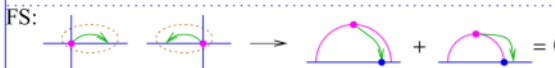
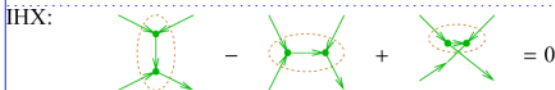
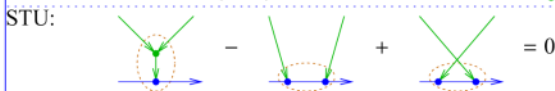
Configuration Space Integrals.



The "Partition Function" Z .

$$K \mapsto Z(K) := \sum_{\text{Feynman diagrams } C(D)} \int \Phi_D^*(\omega_1^{\wedge e(D)}) \in \mathcal{A}^l := \langle D \rangle / (\partial\text{-relations}).$$

∂ -relations. STU, IHX, Foot Swap (FS), Arrow Exchange (AE), and Combinatorial R2 (CR2):



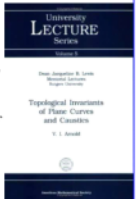
An unfinished project

- * We don't know if T is injective.
- * We don't know if all of \mathcal{A}^l is necessary - it is very possible that it is enough to restrict to the green-less part of \mathcal{A}^l - to "Gauss diagram formulas".
- * A few further configuration space integrals can be written beyond those we have used. We don't know what to do with those, if anything.
- * We don't know the relationship, if any, with algebra.
- * -//- with QFT.
- * We don't know how to do similar things 2-knots

* Nothing is written up.

References. The root, of course, is [Ar]. Further references on doodles include [Kh, FT, Me, Ta, Va1, Va2]. On Goussarov finite-type: [Go, BN].

[Ar] V.I. Arnold, *Topological Invariants of Plane Curves and Caustics*, American Mathematical Society, 1994.
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 [Kh] M. Khovanov, *Doodle Groups*, *Trans. Amer. Math. Soc.* 349-6 (1997) 2297-2315.
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 [Ta] S. Tabachnikov, *Invariants of Smooth Triple Point Free Plane Curves*, *Jour. of Knot Theory and its Ramifications* 5-4 (1996) 531-552.
 [Va1] V.A. Vassiliev, *On Finite Order Invariants of Triple Point Free Plane Curves*, 1999 preprint, [arXiv:1407.7227](https://arxiv.org/abs/1407.7227).
 [Va2] V.A. Vassiliev, *Invariants of Ornaments*, *Adv. in Soviet Math.* 21 (1994) 225-262.



"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

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