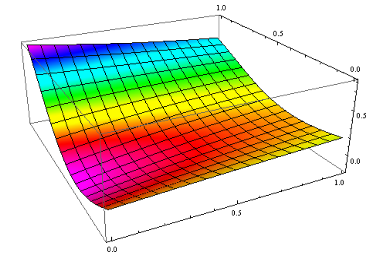
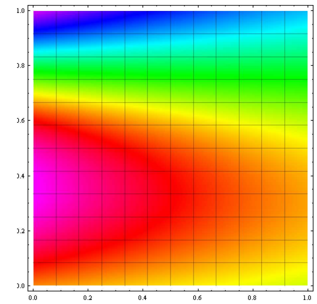
Dessert: Hilbert's 13th Problem, in Full Colour

Dror Bar-Natan, Toronto November 2014. More at
http://www.math.toronto.edu/~drorbn/Talks/Fields-1411

Abstract. To break a week of deep thinking with a nice colourful light dessert, we will present the Kolmogorov-Arnold solution of Hilbert's 13th problem with lots of computer-generated rainbow-painted 3D pictures.
In short, Hilbert asked if a certain specific function of three variables can be written as a multiple (yet finite) composition of continuous functions of just two variables. Kolmogorov and Arnold showed him silly (ok, it took about 60 years, so it was a bit tricky) by showing that **any** continuous function f of any finite number of variables is a finite composition of continuous functions of a single variable and several instances of the binary function "+" (addition). For $f(x,y)=xy$, this may be $xy=\exp(\log x+\log y)$. For $f(x,y,z)=x^y/z$, this may be $\exp(\exp(\log y+\log\log x)+(-\log z))$. What might it be for (say) the real part of the Riemann zeta function?
The only original material in this talk will be the pictures; the math was known since around 1957.

$\frac{1}{3}\mathrm{Re}(\zeta(x+iy))$ on $[0,1]\times[13,17]$

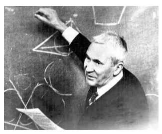
Fix an irrational $\lambda>0$, say $\lambda=(\sqrt{5}-1)/2$. All functions are continuous.

Theorem. There exist five $\phi_i:[0,1]\to[0,1]$ $(1\le i\le 5)$ so that for every $f:[0,1]\times[0,1]\to\mathbb{R}$ there exists a $g:[0,1+\lambda]\to\mathbb{R}$ so that

$$f(x,y)=\sum_{i=1}^{5} g(\phi_i(x)+\lambda\phi_i(y))$$

for every $x,y\in[0,1]$.

Hilbert

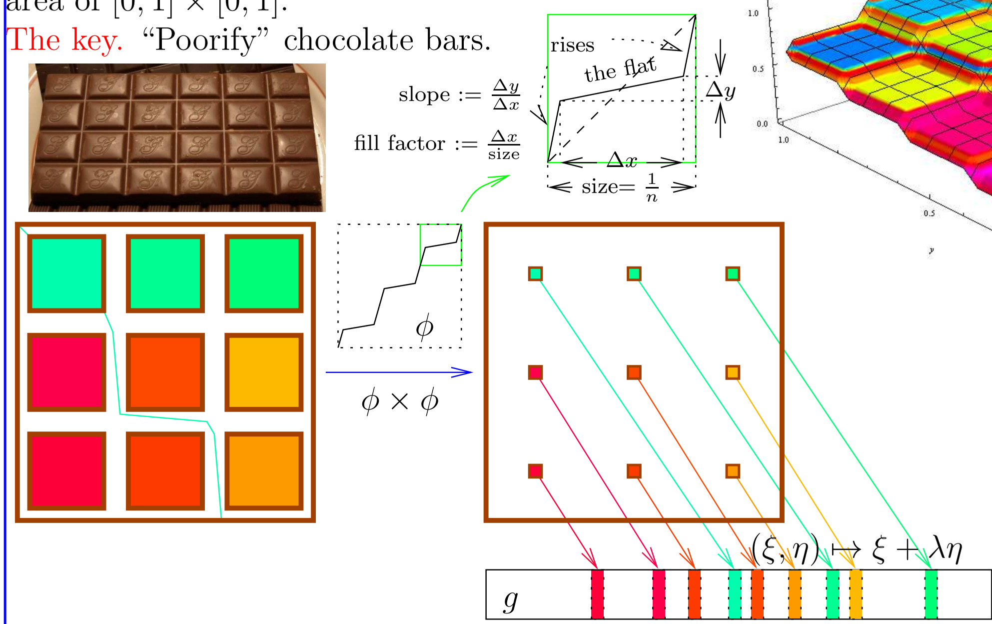


Kolmogorov

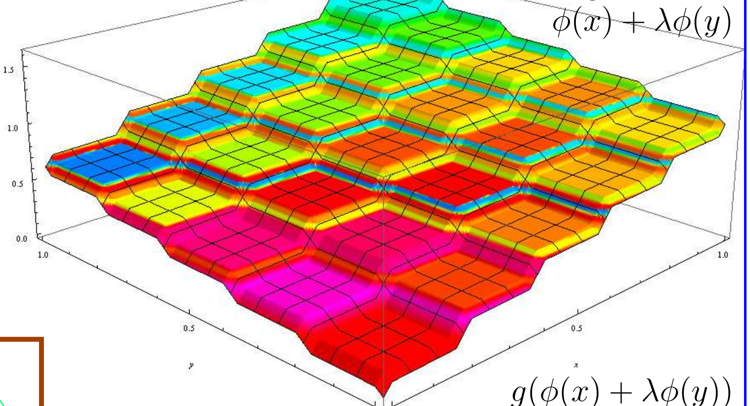
Arnold (by Moser)

Step 1. If $\epsilon>0$ and $f:[0,1]\times[0,1]\to\mathbb{R}$, then there exists $\phi:[0,1]\to[0,1]$ and $g:[0,1+\lambda]\to\mathbb{R}$ so that $|f(x,y)-g(\phi(x)+\lambda\phi(y))|<\epsilon$ on at least 98% of the area of $[0,1]\times[0,1]$.

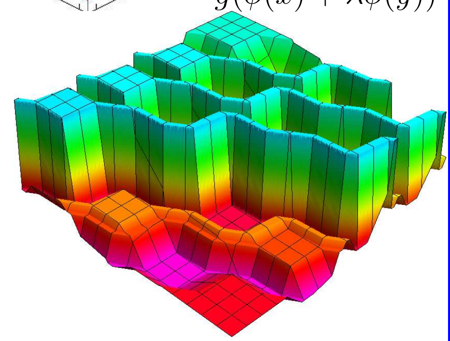
The key. "Poorify" chocolate bars.

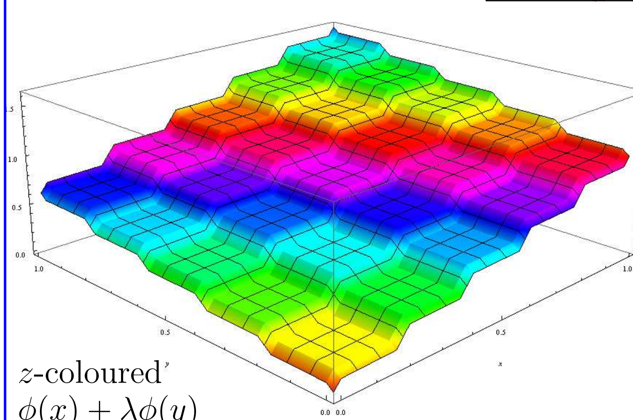


g-coloured $\phi(x)+\lambda\phi(y)$

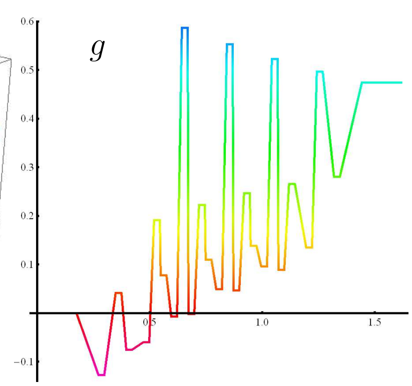


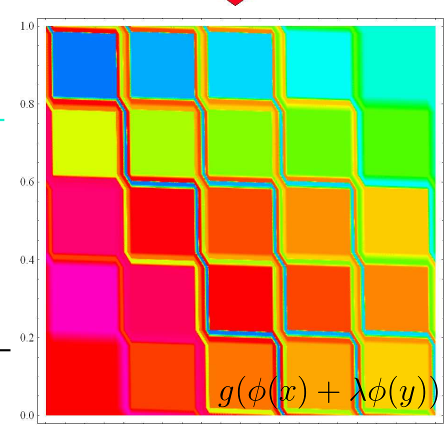
$g(\phi(x)+\lambda\phi(y))$





z-coloured $\phi(x)+\lambda\phi(y)$





$g(\phi(x)+\lambda\phi(y))$

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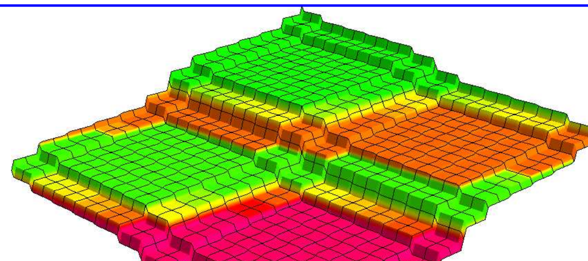
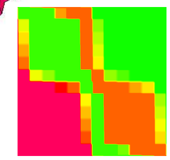
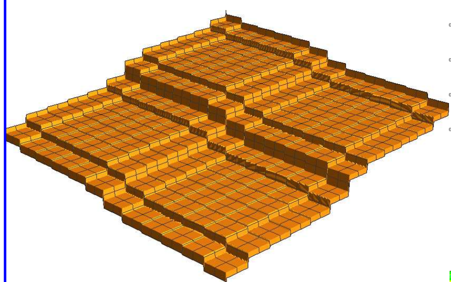




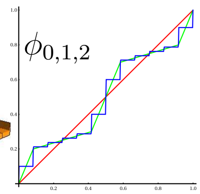
g_1-coloured $\phi(x)+\lambda\phi(y)$

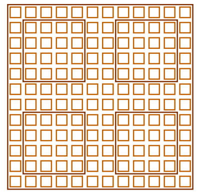
Step 2. There exists $\phi : [0,1] \to [0,1]$ so that for every $\epsilon > 0$ and every $f : [0,1]\times[0,1] \to \mathbb{R}$ there exists a $g : [0,1+\lambda] \to \mathbb{R}$ so that $|f(x,y) - g(\phi(x)+\lambda\phi(y))| < \epsilon$ on a set of area at least $1-\epsilon$ in $[0,1]\times[0,1]$.

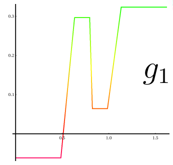
The key. "Iterated poorification".

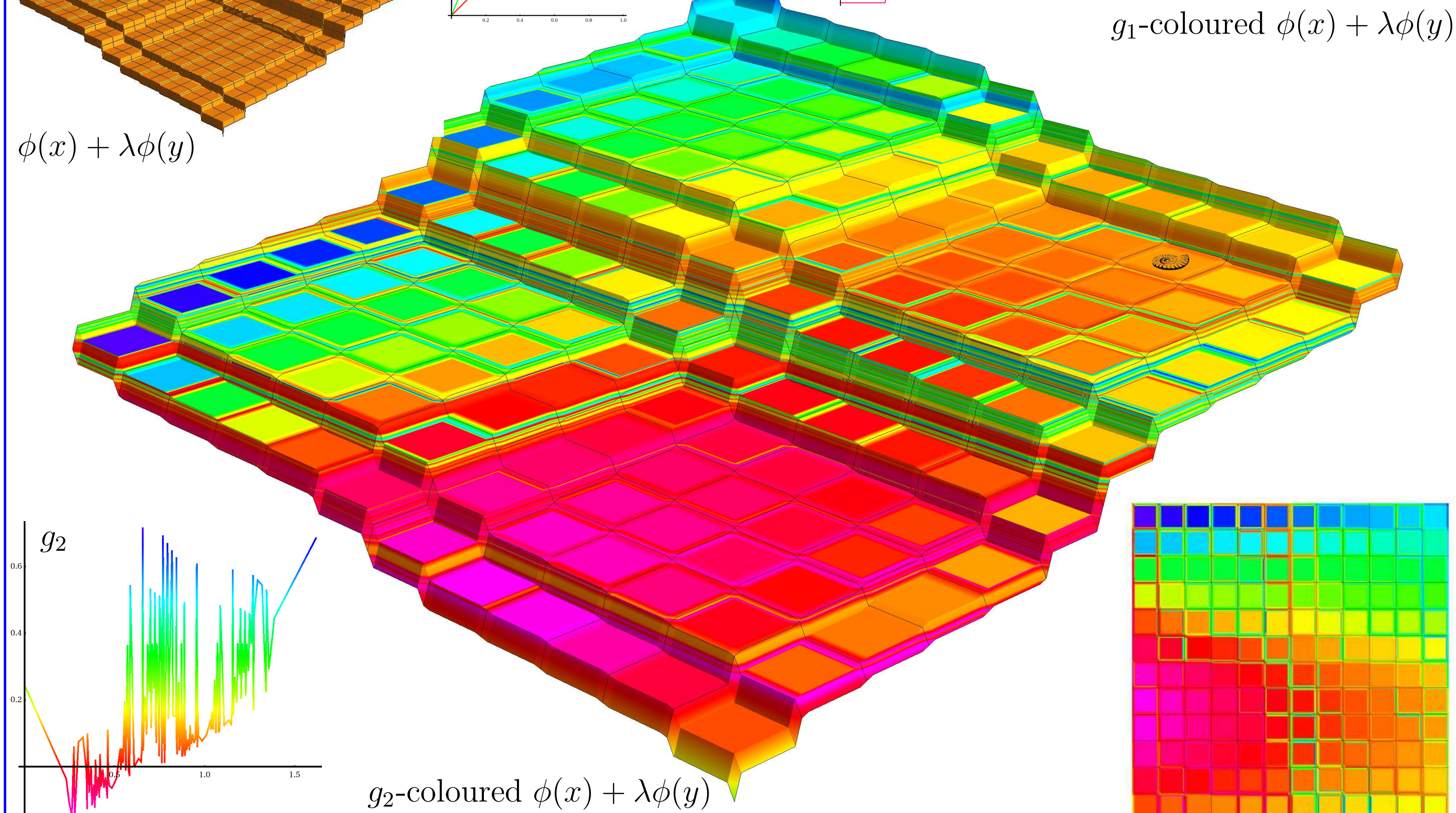


$\phi(x)+\lambda\phi(y)$









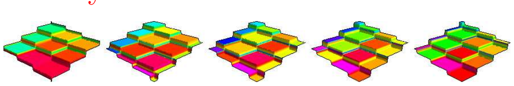
g_2-coloured $\phi(x)+\lambda\phi(y)$

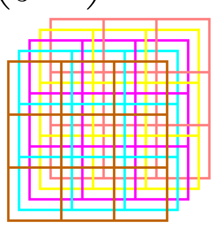
Step 3. There exist $\phi_i : [0,1] \to [0,1]$ $(1 \le i \le 5)$ so that for every $\epsilon > 0$ and every $f : [0,1]\times[0,1] \to \mathbb{R}$ there exists a $g : [0,1+\lambda] \to \mathbb{R}$ so that

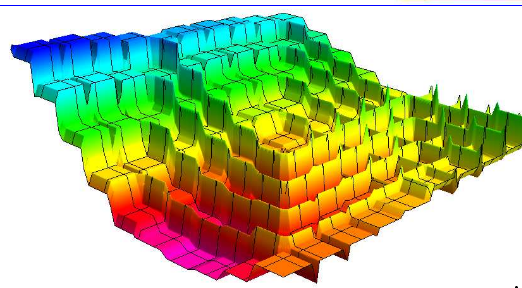
$$\left|f(x,y) - \sum_{i=1}^{5} g(\phi_i(x)+\lambda\phi_i(y))\right| < \left(\frac{2}{3}+\epsilon\right)\|f\|_\infty$$

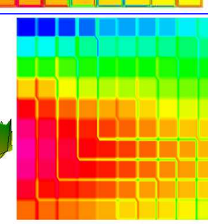
for every $x, y \in [0,1]$.

The key. "Shift the chocolates"...









...then iterate.

Step 4. We are done.

The key. Learn from the artillery!

Set $Tg := \sum_{i=1}^{5} g(\phi_i(x)+\lambda\phi_i(y))$, $f_1 := f$, $M := \|f\|$, and iterate "shooting and adjusting". Find g_1 with $\|g_1\| \le M$ and $\|f_2 := f_1 - Tg_1\| \le \frac{3}{4}M$. Find g_2 with $\|g_2\| \le \frac{3}{4}M$ and $\|f_3 := f_2 - Tg_2\| \le (\frac{3}{4})^2 M$. Find g_3 with $\|g_3\| \le (\frac{3}{4})^2 M$ and $\|f_4 := f_3 - Tg_3\| \le (\frac{3}{4})^3 M$. Continue to eternity. When done, set $g = \sum g_k$ and note that $f = Tg$ as required.

Exercise 1. Do the m-dimensional case.

Exercise 2. Do $\mathbb{R}^m$ instead of just I^m.

Propaganda. I love handouts! • I have nothing to hide and you can take what you want, forwards, backwards, here and at home. • What doesn't fit on one sheet can't be done in one hour. • It takes learning and many hours and a few pennies. The audience's worth it! • There's real math in the handout viewer!