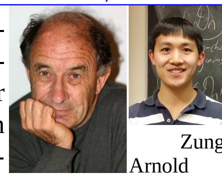
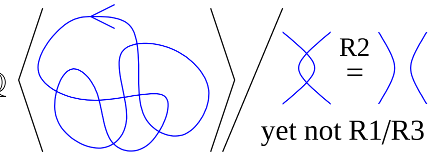
Finite Type Invariants of Doodles, 1

Abstract. I will describe my former student's Jonathan Zung work on finite type invariants of "doodles", plane curves modulo the second Reidemeister move but not modulo the third. We use a definition of "finite type" different from Arnold's and more along the lines of Goussarov's "Interdependent Modifications", and come to a conjectural combinatorial description of the set of all such invariants. We then describe how to construct many such invariants (though perhaps not all) using a certain class of 2-dimensional "configuration space integrals".

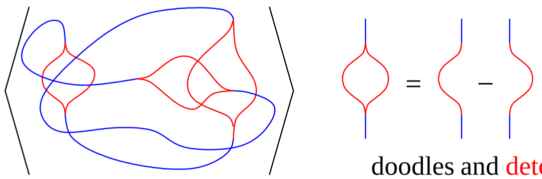

Arnold Zung

Doodles.

$\mathcal{K} = \mathbb{Q}$ 

yet not R1/R3

Goussarov Finite-Type.

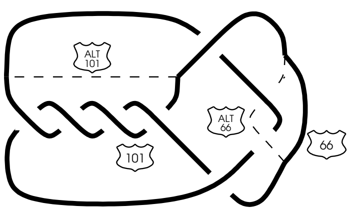
$\mathcal{K}_n =$ 


Goussarov

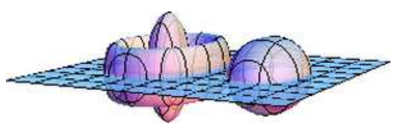
doodles and detours (dnd's)

Definition. V is of type n if it vanishes on $\mathcal{K}_{n+1}$.

Knots in 3D.

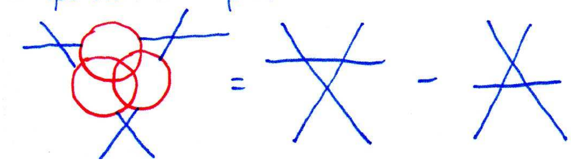


2-Knots in 4D.



The reason I care!

Important Example:



$\mathcal{D}^c/\text{2NT} = \mathcal{A}^c \twoheadrightarrow \text{gr}K \xrightarrow{\text{gr}2} \mathcal{A}^t$, $\mathcal{A}^c \xrightarrow{T} \mathcal{A}^t$, $\mathcal{A}^t \xrightarrow{?\ \text{projection map}} \mathcal{A}^c$

Jonathan's Comment. It seems that the configuration space integrals we defined are more naturally invariants of virtual doodles. Virtual doodles are doodles with some ordinary crossings and some virtual crossings, with the relation that triple points having three virtual crossings are allowed. (Caution: virtual doodles are not Gauss diagrams modulo Reidemeister 2.)

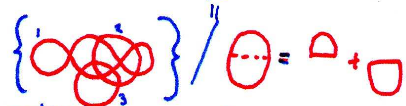
The integrals we defined are invariants for virtual doodles, if we use the rule that the Gauss diagram skeleton is not allowed to use virtual crossings.

What kinds of chords do our integrals detect? They detect "semi-virtuals with outer rings".

(Conjectured) Punchline: Relations on Feynman diagrams correspond with relations on chord diagrams. This is just a matter of carefully checking the analogues of the relations we already knew. What makes this work here and not in the original theory is that we have degree 2 chords, the semi-virtuals.

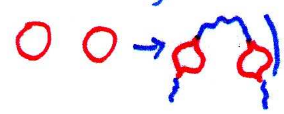
What would be nice is a clean formulation of finite type for virtual doodles yielding chords which are "semi-virtuals with outer rings".

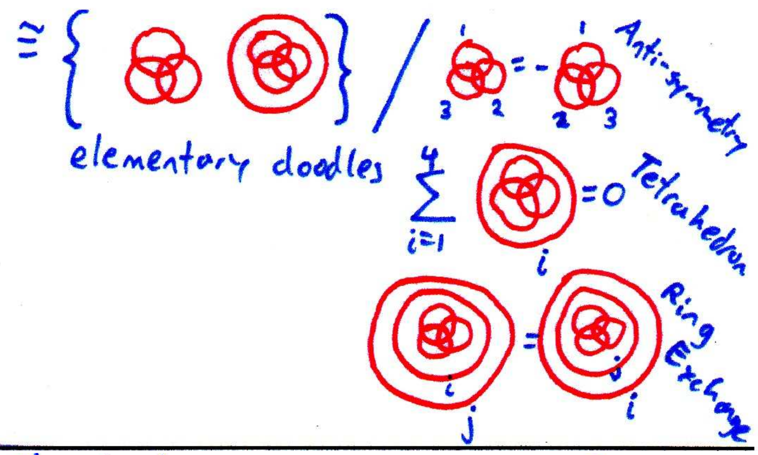
Upper bound on K_n/K_{n+1} $\mathcal{D}^c$

$K_n/K_{n+1} \twoheadleftarrow$ 

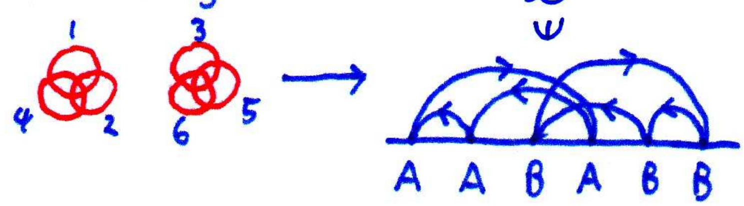
ordered n-component doodles, +1 winding number

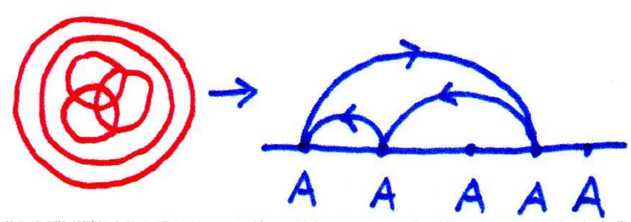
(Modulo K_{n+1}, all ways of connecting bubbles are equivalent.



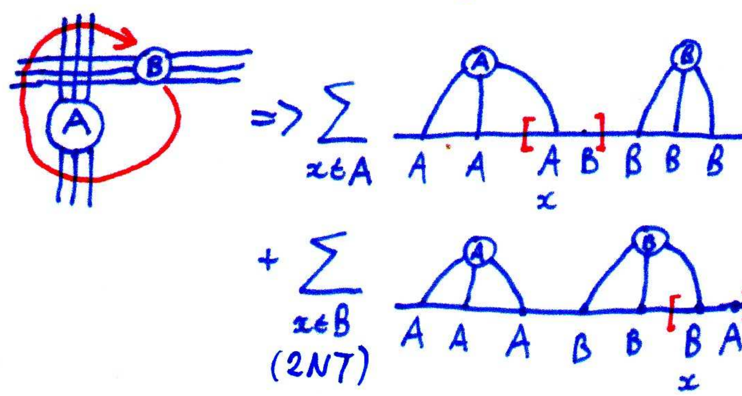


Chord diagrams





Relations on Chord Diagrams



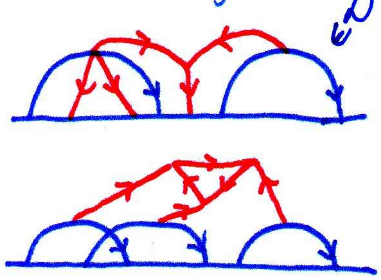
Arrow diagrams

$\mathcal{D}^t$

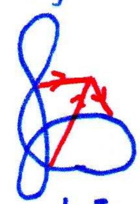
Can also have blues w/ no emanating reds.

Gauss diagram skeleton.

2-in-1-out internal vertices Why?



Configuration space integrals



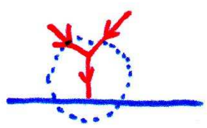
$$Z_{guess} = \sum_{\substack{\text{arrow}\\ \text{diagrams}\\ D}} \oint_{Conf(D)} \Phi^*(\omega')^{\#\text{arrows}} [D]$$

$\downarrow \Phi$

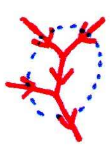
$(\rightarrow, \nearrow, \searrow) \in (S^1)^3$

$\mathcal{A}^t = \mathcal{D}^t /$ STU, IHX, foot swap, arrow exchange, R2 invariance

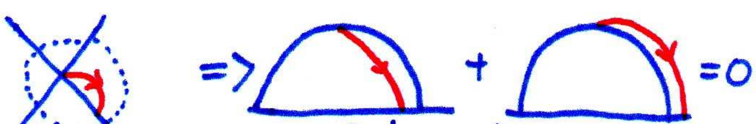
Primary Faces => Relations on Arrow Diagrams



=> STU

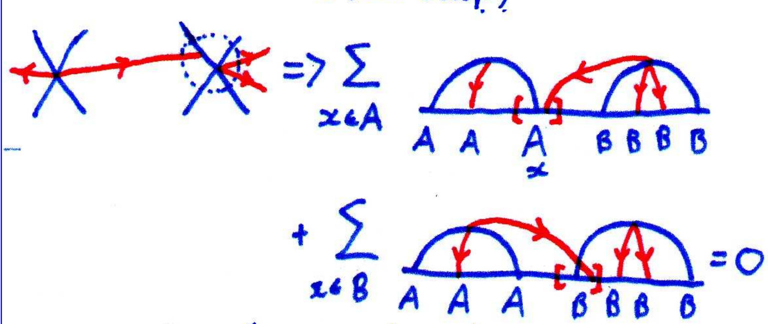


=> IHX



$\Rightarrow$ + = 0

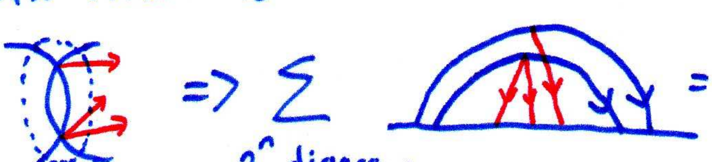
(foot swap)



$\Rightarrow \sum_{x\in A}$ A A A B B B B + $\sum_{x\in B}$ A A A B B B B = 0

(arrow exchange)

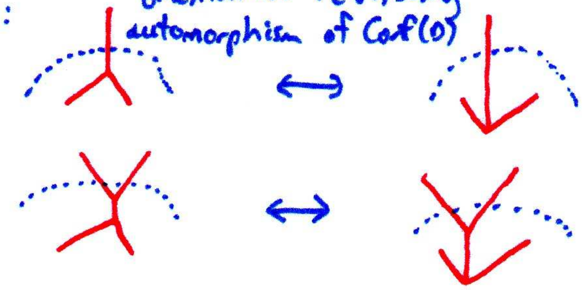
R2 invariance



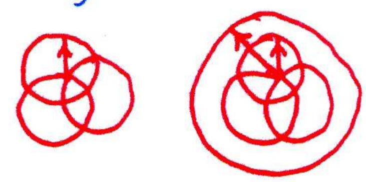
$\Rightarrow \sum_{2^n \text{ diagrams}}$ = 0

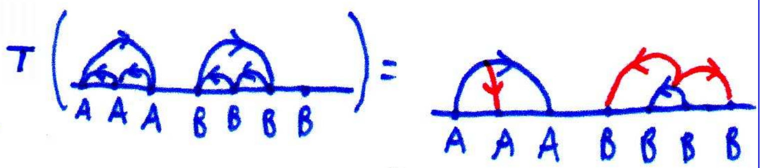
Hidden faces vanish

Usual trick: orientation reversing automorphism of Conf(D)



Evaluating Z on n-bracelets

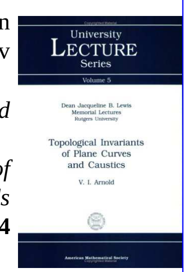




T(A A A B B B B) = A A A B B B B

+ 8 other diagrams

References. The root, of course, is [Ar]. Further references on doodles include [Kh, FT, Me, Ta, Va1, Va2]. On Goussarov finite-type: [Go, BN].



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[Va2] V.A. Vassiliev, *Invariants of Ornaments*, Adv. in Soviet Math. **21** (1994) 225–262.

"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

→ Safekeeping / Recycling. / Notes

where does Arnolds strangeness fit in?

Perhaps I should put in the Markov construction?

Nearby objects: "virtual doodles", doodles w/ dots, planar graphs of various kinds, ...