Thang Le on the Habiro Ring Etc.

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Jones: $J : \{\text{oriented links}\} \longrightarrow \mathbb{Z}[q^{\pm 1/2}]$ satisfies

$$q \,\nearrow\!\!\!\!\nwarrow - q^{-1} \,\nwarrow\!\!\!\!\nearrow = (q^{1/2} - q^{-1/2}) \uparrow\uparrow$$

and

$$O \sqcup L = [2]\, L \quad \text{where} \quad [n] = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}$$

and

$$J(\emptyset) = 1$$

uniqueness is easy, existence not too hard.

Generalizations: .... If $\mathfrak{g}$ is a simple Lie algebra, K is an oriented framed knot, V a f.d. $\mathfrak{g}$-module $\Rightarrow$

$$J_K(V, q) \in \mathbb{Z}[q^{\pm 1/d}] \quad d = d(\mathfrak{g}) = 2\det \operatorname{cartan}(\mathfrak{g})$$

Example: $\mathfrak{g} = sl_2$, $V = \mathbb{C}^2 \Rightarrow$ Jones.

Witten: 1988: invts of pairs (M^3, L)

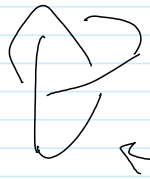
Reshetikhin-Turaev 1989: a rigorous def.

At the end: if $\mathfrak{g}$ is a simple Lie alg, M a closed 3D oriented manifold, $\xi \in Z_{\mathfrak{g}}$ (roots of unity w/ some restriction on the order)

$$\longrightarrow WRT^{\mathfrak{g}}_M(\xi) \in \mathbb{C}.$$

<u>Example</u> $M = \Sigma(2,3,5)$ $WRT^{sl_2}_M(q) =$ complicated formula.

Kashaev invt:



$$\overbrace{K_K(q)}^{\text{Kashaev}} = \sum_{n=0}^{\infty} (1-q)(1-q^2)\cdots(1-q^n)$$

What type of functions have we got?

<u>Conjecture</u> (Kont., Lawrence) $WRT^{\mathfrak{g}}_M(\xi)$ is always

an alg. integer.

The Habiro Ring: $\mathbb{Z} \subset R \subset \mathbb{Q}$ [all invts take values there]

$$\widehat{R[q]} = \varprojlim_n R[q]/(q)_n \qquad (q)_n = (1-q)(1-q^2)\cdots(1-q^n)$$

The Habiro ring is $\widehat{\mathbb{Z}[q]}$ "analytic functions on roots of 1".

$$f \in \widehat{\mathbb{Z}[q]} \Rightarrow f = \sum_{n=0}^{\infty} f_n(q)(q)_n \qquad f_n(q) \in \mathbb{Z}[q]$$

$$(q)_{n+1} = (1-q^{n+1})(q)_n = (q)_n - q^{n+1}(q)_n$$

Related to F_1!

$$\Rightarrow \quad 1 = \sum q^{n+1}(q)_n$$

The presentation $f = \sum f_n (q)_n$ is unique if $\deg f_n \leq n$

Properties: 0. $f \in \widehat{\mathbb{Z}[q]}$ then f defines

$$f : \underset{\substack{\uparrow \\ \text{roots of } 1.}}{\mathbb{Z}} \to \mathbb{C}$$

1. $$f' = \sum_{n=0}^{\infty} f_n'(q)(q)_n + f_n(q)\underbrace{(q)_n'}_{\text{divisible by } (q)_{\lfloor \frac{n+1}{2} \rfloor}}$$

$$\Rightarrow f' \in \widehat{\mathbb{Z}[q]}$$

$$T_\xi f = \sum_{n=0}^{\infty} \frac{f^{(n)}(\xi)}{n!}(q-\xi)^n \in \mathbb{Z}[\xi][[q-\xi]]$$

.... also a nice Taylor expansion at $q=1$.

Thm (Habiro, Vogel) $f, g \in \widehat{\mathbb{Z}[q]}$ if their Taylor expansions at some point are the same, they are the same. In particular:

$T_1 : \widehat{\mathbb{Z}[q]} \to \mathbb{Z}[[(1-q)]]$ is an injection. In particular, $\widehat{\mathbb{Z}[q]}$ is a domain

not surjective!

Replacing $\mathbb{Z}$ by $\mathbb{Q}$ $T_1 : \widehat{\mathbb{Q}[q]} \to \mathbb{Q}[[1-q]]$ is surjective but not injective.

2. Thm (Habiro) $f, g \in \widehat{\mathbb{Z}[q]}$ suppose $f(\zeta) = g(\zeta)$ on some set Z' of roots of 1 that has a limit point in Z, then $f = g$.

"limit pt" in the "cyclotomic topology".

<u>Example</u> $\zeta_k = e^{2\pi i/k}$ $Z' = \{\zeta_{7^k} : k \in \mathbb{N}\}$

$$Z' = \{\zeta_p : p \in \text{primes}\}$$

$\Rightarrow f \in \widehat{\mathbb{Z}[q]}$ means f is totally determined by $f(\zeta_{7^k})$

Continued July 9:

A relation w/ the p-adic ring: p: prime

$$\mathbb{Z}_p = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z} = \left\{\sum_{n=0}^{\infty} a_n p^n : a_n \in \mathbb{Z}\right\}$$

If p is an odd prime & $f \in \widehat{\mathbb{Z}[q]}$: $P / Q^{p-1} - 1$

so $f(2)$ makes sense in $\mathbb{Z}_p$

In fact, $\widehat{\mathbb{Z}[q]}\big|_{q=2} = \prod_{p \text{ odd prime}} \mathbb{Z}_p$

$\mathfrak{g}$: simple Lie alg, M: oriented 3-manifold,

ξ: a root of unity satisfying some conditions.

[$Z_{\mathfrak{g}}$: the set of relevant $\sqrt{1}$'s]

<u>Thm</u> If M is an $\mathbb{Z}HS$, the $\exists!$ $J_M^{\mathfrak{g}} \in \widehat{\mathbb{Z}[q]}$

s.t. $\forall \xi \in Z_{\mathfrak{g}}$ $J_M^{\mathfrak{g}}(\xi) = WRT_M^{\mathfrak{g}}(\xi)$

<u>Thm</u> For every knot $K \subset S^3$ $\exists$ $I_K \in \widehat{\mathbb{Z}[q]}$ s.t.

$\forall \xi$ $I_K(\xi_n) = K_K(\xi_n)$ $\xi_n = e^{2\pi i/n}$

Kashaev.

Surgery & the Kirby moves.

The construction of the RT invariants.

$V_n = Sym^{(n-1)} V$ $V = \mathbb{C}^2$ $\mathfrak{g} = sl_2$

The "Kirby Colour" is .....

<u>"Pseudo Thm"</u> $F_L(\xi) = \sum_{n_1 \dots n_k \in Z_{\mathfrak{g}}} J_K(n_1 \dots n_k)[n_1] \dots [n_k]$

is invariant under Kirby I.

But we need to regularize the RHS....

⋮

then there is fixing K-II.

Thm "Commutes w/ LMO".