```
U[i_, j_] := ReplacePart[IdentityMatrix[3], {
   {i, i} → 1, {i, j} → 0,
   {j, i} → 1 - t_j, {j, j} → t_i
  }]
```

```
U[1, 2]〚1 ;; 2, 1 ;; 2〛 // MatrixForm
```

$$\begin{pmatrix} 1 & 0 \\ 1 - t_2 & t_1 \end{pmatrix}$$

```
MatrixForm /@ Simplify /@ {U[1, 2].U[1, 3].U[2, 3], U[2, 3].U[1, 3].U[1, 2]}
```

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 1 - t_2 & t_1 & 0 \\ 1 - t_3 & -t_1 (-1 + t_3) & t_1 t_2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 - t_2 & t_1 & 0 \\ 1 - t_3 & -t_1 (-1 + t_3) & t_1 t_2 \end{pmatrix} \right\}$$

```
Eigensystem[U[1, 2]〚1 ;; 2, 1 ;; 2〛]
```

$$\left\{ \{1, t_1\}, \left\{ \left\{ -\frac{1 - t_1}{-1 + t_2}, 1 \right\}, \{0, 1\} \right\} \right\}$$

```
Eigensystem[U[1, 2]〚1 ;; 2, 1 ;; 2〛 // Transpose]
```

$$\left\{ \{1, t_1\}, \left\{ \{1, 0\}, \left\{ -\frac{-1 + t_2}{-1 + t_1}, 1 \right\} \right\} \right\}$$

```
Inverse[( 1 - t_1   0  )] . ( 1     0  ) . ( 1 - t_1   0  ) // Simplify // MatrixForm
        (   0    1 - t_2 )   ( 1 - t_2  t_1 )   (   0    1 - t_2 )
```

$$\begin{pmatrix} 1 & 0 \\ 1 - t_1 & t_1 \end{pmatrix}$$

```
V[i_, j_] := ReplacePart[IdentityMatrix[3], {
   {i, i} → 1, {i, j} → 0,
   {j, i} → 1 - x, {j, j} → t_i
  }]
```

```
V[1, 2]〚1 ;; 2, 1 ;; 2〛 // MatrixForm
```

$$\begin{pmatrix} 1 & 0 \\ 1 - x & t_1 \end{pmatrix}$$

```
MatrixForm /@ Simplify /@ {V[1, 2].V[1, 3].V[2, 3], V[2, 3].V[1, 3].V[1, 2]}
```

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 1 - x & t_1 & 0 \\ 1 - x & -(-1 + x) t_1 & t_1 t_2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 - x & t_1 & 0 \\ (-1 + x) (-1 + x - t_2) & -(-1 + x) t_1 & t_1 t_2 \end{pmatrix} \right\}$$

```
Eigenvalues[V[1, 2]〚1 ;; 2, 1 ;; 2〛]
```

$$\{1, t_1\}$$

```
Solve[1 - x == (-1 + x) (-1 + x - t_2), x]
```

$$\{\{x \to 1\}, \{x \to t_2\}\}$$

```
U[i_, j_] := ReplacePart[IdentityMatrix[3], {
    {i, i} → 1, {i, j} → 0,
    {j, i} → 1 - α tᵢ - β tⱼ, {j, j} → tᵢ
  }]
```

```
U[1, 2][[1 ;; 2, 1 ;; 2]] // MatrixForm
```

$$\begin{pmatrix} 1 & 0 \\ 1 - \alpha t_1 - \beta t_2 & t_1 \end{pmatrix}$$

```
MatrixForm /@ Simplify /@ {U[1, 2].U[1, 3].U[2, 3], U[2, 3].U[1, 3].U[1, 2]}
```

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 1 - \alpha t_1 - \beta t_2 & t_1 & 0 \\ 1 - \alpha t_1 - \beta t_3 & -t_1 (-1 + \alpha t_2 + \beta t_3) & t_1 t_2 \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} 1 & 0 & 0 \\ 1 - \alpha t_1 - \beta t_2 & t_1 & 0 \\ -t_2 (-1 + \alpha t_1 + \beta t_3) + (-1 + \alpha t_1 + \beta t_2) (-1 + \alpha t_2 + \beta t_3) & -t_1 (-1 + \alpha t_2 + \beta t_3) & t_1 t_2 \end{pmatrix} \right\}$$

```
Eigenvalues[U[1, 2][[1 ;; 2, 1 ;; 2]]]
```

$\{1, t_1\}$

```
SolveAlways[
  1 - α t₁ - β t₃ == -t₂ (-1 + α t₁ + β t₃) + (-1 + α t₁ + β t₂) (-1 + α t₂ + β t₃), {t₁, t₂, t₃}]
```

$\{\{\alpha \to 0, \beta \to 1\}, \{\alpha \to 1, \beta \to 0\}\}$