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U[i_, j_] := ReplacePart[IdentityMatrix[3], {
  {i, i} → α₀ + α₁ ti + α₂ tj, {i, j} → β₀ + β₁ ti + β₂ tj,
  {j, i} → γ₀ + γ₁ ti + γ₂ tj, {j, j} → δ₀ + δ₁ ti + δ₂ tj
}]
}

U[1, 2].U[1, 3].U[2, 3] // MatrixForm

(
  (α₀ + t1 α1 + t2 α2) (α₀ + t1 α1 + t3 α2) (α₀ + t2 α1 + t3 α2) (β₀ + t1 β1 + t2 β2) + (α₀ + t1 α1 + t2 α2) (β₀ + t1 β1 + t3 β2) + (γ₀ + t1 γ1 + t2 γ2) (γ₀ + t1 γ1 + t3 γ2) + (γ₀ + t2 γ1 + t3 γ2) (δ₀ + t1 δ1 + t3 δ2)
  (α₀ + t1 α1 + t3 α2) (γ₀ + t1 γ1 + t2 γ2) (β₀ + t1 β1 + t3 β2) (γ₀ + t1 γ1 + t2 γ2) + (γ₀ + t2 γ1 + t3 γ2) (γ₀ + t1 γ1 + t3 γ2) + (γ₀ + t2 γ1 + t3 γ2) (δ₀ + t1 δ1 + t2 δ2)
  γ₀ + t1 γ1 + t3 γ2 (γ₀ + t2 γ1 + t3 γ2) (δ₀ + t1 δ1 + t2 δ2) (δ₀ + t1 δ1 + t3 δ2)
)

U[2, 3].U[1, 3].U[1, 2] // MatrixForm

(
  (α₀ + t1 α1 + t2 α2) (α₀ + t1 α1 + t3 α2) (α₀ + t2 α1 + t3 α2) (γ₀ + t1 γ1 + t2 γ2) + (α₀ + t1 α1 + t2 α2) (β₀ + t2 β1 + t3 β2) (γ₀ + t1 γ1 + t3 γ2) + (γ₀ + t1 γ1 + t2 γ2) (γ₀ + t2 γ1 + t3 γ2) + (α₀ + t1 α1 + t2 α2) (γ₀ + t1 γ1 + t2 γ2) + (α₀ + t1 α1 + t2 α2) (γ₀ + t1 γ1 + t3 γ2) + (δ₀ + t2 δ1 + t3 δ2) (δ₀ + t1 δ1 + t3 δ2)
  (α₀ + t2 α1 + t3 α2) (γ₀ + t1 γ1 + t2 γ2) + (α₀ + t1 α1 + t2 α2) (β₀ + t2 β1 + t3 β2) (γ₀ + t1 γ1 + t3 γ2) + (γ₀ + t1 γ1 + t2 γ2) (γ₀ + t2 γ1 + t3 γ2) + (α₀ + t1 α1 + t2 α2) (β₀ + t1 β1 + t3 β2) (δ₀ + t2 δ1 + t3 δ2) = β₀ + t1 β1 + t3 β2 &&
  (α₀ + t1 α1 + t3 α2) (γ₀ + t1 γ1 + t2 γ2) = (α₀ + t2 α1 + t3 α2) (γ₀ + t1 γ1 + t2 γ2) + (α₀ + t1 α1 + t2 α2) (β₀ + t2 β1 + t3 β2) (γ₀ + t1 γ1 + t3 γ2) &&
  (α₀ + t1 α1 + t2 α2) (β₀ + t1 β1 + t3 β2) (γ₀ + t1 γ1 + t3 γ2) &&
  (β₀ + t1 β1 + t3 β2) (γ₀ + t1 γ1 + t2 γ2) (γ₀ + t2 γ1 + t3 γ2) &&
  (β₀ + t2 β1 + t3 β2) (δ₀ + t1 δ1 + t2 δ2) + (β₀ + t1 β1 + t3 β2) (γ₀ + t1 γ1 + t2 γ2) (δ₀ + t2 δ1 + t3 δ2) = δ₀ + t2 δ1 + t3 δ2
  (β₀ + t2 β1 + t3 β2) (δ₀ + t1 δ1 + t3 δ2) &&
  γ₀ + t1 γ1 + t3 γ2 = (γ₀ + t1 γ1 + t2 γ2) (γ₀ + t2 γ1 + t3 γ2) + (α₀ + t1 α1 + t2 α2) (γ₀ + t1 γ1 + t3 γ2) (δ₀ + t2 δ1 + t3 δ2) &&
  (γ₀ + t2 γ1 + t3 γ2) (δ₀ + t1 δ1 + t3 δ2) = (γ₀ + t2 γ1 + t3 γ2) (δ₀ + t1 δ1 + t2 δ2) + (β₀ + t1 β1 + t2 β2) (γ₀ + t1 γ1 + t3 γ2) (δ₀ + t2 δ1 + t3 δ2)
)

eqns = Simplify[And @@ Thread[Flatten[U[1, 2].U[1, 3].U[2, 3]] == Flatten[U[2, 3].U[1, 3].U[1, 2]]]]
(α₀ + t2 α1 + t3 α2) (β₀ + t1 β1 + t2 β2) + (α₀ + t1 α1 + t2 α2) (β₀ + t1 β1 + t3 β2) (γ₀ + t2 γ1 + t3 γ2) =
  (α₀ + t1 α1 + t3 α2) (β₀ + t1 β1 + t2 β2) && (β₀ + t1 β1 + t2 β2) (β₀ + t2 β1 + t3 β2) +
  (α₀ + t1 α1 + t2 α2) (β₀ + t1 β1 + t3 β2) (δ₀ + t2 δ1 + t3 δ2) = β₀ + t1 β1 + t3 β2 &&
  (α₀ + t1 α1 + t3 α2) (γ₀ + t1 γ1 + t2 γ2) = (α₀ + t2 α1 + t3 α2) (γ₀ + t1 γ1 + t2 γ2) + (α₀ + t1 α1 + t2 α2) (β₀ + t2 β1 + t3 β2) (γ₀ + t1 γ1 + t3 γ2) &&
  (β₀ + t1 β1 + t3 β2) (β₀ + t2 β1 + t3 β2) (γ₀ + t1 γ1 + t3 γ2) &&
  (β₀ + t2 β1 + t3 β2) (δ₀ + t1 δ1 + t2 δ2) + (β₀ + t1 β1 + t3 β2) (γ₀ + t1 γ1 + t2 γ2) (δ₀ + t2 δ1 + t3 δ2) =
  (β₀ + t2 β1 + t3 β2) (δ₀ + t1 δ1 + t3 δ2) &&
  γ₀ + t1 γ1 + t3 γ2 = (γ₀ + t1 γ1 + t2 γ2) (γ₀ + t2 γ1 + t3 γ2) + (α₀ + t1 α1 + t2 α2) (γ₀ + t1 γ1 + t3 γ2) (δ₀ + t2 δ1 + t3 δ2) &&
  (γ₀ + t2 γ1 + t3 γ2) (δ₀ + t1 δ1 + t3 δ2) = (γ₀ + t2 γ1 + t3 γ2) (δ₀ + t1 δ1 + t2 δ2) + (β₀ + t1 β1 + t2 β2) (γ₀ + t1 γ1 + t3 γ2) (δ₀ + t2 δ1 + t3 δ2)
)

sols0 = Union[Union /@ SolveAlways[eqns, {t1, t2, t3}]]
$Aborted

sols1 = Union[Union /@ SolveAlways[eqns && (α₀ + α1 + α2 == 1) &&
  (β₀ + β1 + β2 == 0) && (γ₀ + γ1 + γ2 == 0) && (δ₀ + δ1 + δ2 == 1), {t1, t2, t3}]]
$Aborted

sols2 = Union[
  Union /@ SolveAlways[eqns && (α₀ + α1 + α2 == 1) && (β₀ + β1 + β2 == 0) && (γ₀ + γ1 + γ2 == 0) &&
  (δ₀ + δ1 + δ2 == 1) && (α₀ + γ₀ == 1) && (α1 + γ1 == 0) && (α2 + γ2 == 0) &&
  (β₀ + δ0 == 1) && (β1 + δ1 == 0) && (β2 + δ2 == 0), {t1, t2, t3}]]
$Aborted

reds = Solve[(α₀ + α1 + α2 == 1) && (β₀ + β1 + β2 == 0) &&
  (γ₀ + γ1 + γ2 == 0) && (δ₀ + δ1 + δ2 == 1) && (α0 + γ0 == 1) && (α1 + γ1 == 0) &&
  (α2 + γ2 == 0) && (β0 + δ0 == 1) && (β1 + δ1 == 0) && (β2 + δ2 == 0)], // First
{α₀ → 1 + γ1 + γ2, α1 → -γ1, α2 → -γ2,
  β₀ → δ1 + δ2, β1 → -δ1, β2 → -δ2, γ₀ → -γ1 - γ2, δ₀ → 1 - δ1 - δ2}
]

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Simplify[eqns /. reds]
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$$\begin{aligned}
& ((-1 + t_1) \gamma_1 + (-1 + t_2) \gamma_2) (1 - (-1 + t_1) \gamma_1 - (-1 + t_3) \gamma_2) = \\
& \quad ((-1 + t_1) \gamma_1 + (-1 + t_2) \gamma_2) (1 - (-1 + t_2) \gamma_1 - (-1 + t_3) \gamma_2) + \\
& \quad (1 - (-1 + t_1) \gamma_1 - (-1 + t_2) \gamma_2) ((-1 + t_1) \gamma_1 + (-1 + t_3) \gamma_2) ((-1 + t_2) \delta_1 - (-1 + t_3) \delta_2) \&& \\
& (-1 + t_1) \gamma_1 + (-1 + t_3) \gamma_2 = ((-1 + t_1) \gamma_1 + (-1 + t_2) \gamma_2) ((-1 + t_2) \gamma_1 + (-1 + t_3) \gamma_2) + \\
& \quad (1 - (-1 + t_1) \gamma_1 - (-1 + t_2) \gamma_2) \\
& \quad ((-1 + t_1) \gamma_1 + (-1 + t_3) \gamma_2) (1 + (-1 + t_2) \delta_1 + (-1 + t_3) \delta_2) \&& \\
& (-1 + t_1) \gamma_1 + (-1 + t_2) \gamma_2) ((-1 + t_2) \gamma_1 + (-1 + t_3) \gamma_2) (-(-1 + t_1) \delta_1 - (-1 + t_3) \delta_2) = \\
& \quad ((-1 + t_1) \gamma_1 + (-1 + t_3) \gamma_2) (-(-1 + t_1) \delta_1 - (-1 + t_2) \delta_2) (-(-1 + t_2) \delta_1 - (-1 + t_3) \delta_2) \&& \\
& ((-1 + t_2) \gamma_1 + (-1 + t_3) \gamma_2) (1 + (-1 + t_1) \delta_1 + (-1 + t_3) \delta_2) = \\
& \quad ((-1 + t_2) \gamma_1 + (-1 + t_3) \gamma_2) (1 + (-1 + t_1) \delta_1 + (-1 + t_2) \delta_2) + \\
& \quad ((-1 + t_1) \gamma_1 + (-1 + t_3) \gamma_2) (-(-1 + t_1) \delta_1 - (-1 + t_2) \delta_2) (1 + (-1 + t_2) \delta_1 + (-1 + t_3) \delta_2) \&& \\
& (1 - (-1 + t_2) \gamma_1 - (-1 + t_3) \gamma_2) (-(-1 + t_1) \delta_1 - (-1 + t_2) \delta_2) + \\
& \quad (1 - (-1 + t_1) \gamma_1 - (-1 + t_2) \gamma_2) ((-1 + t_2) \gamma_1 + (-1 + t_3) \gamma_2) (-(-1 + t_1) \delta_1 - (-1 + t_3) \delta_2) = \\
& \quad (1 - (-1 + t_1) \gamma_1 - (-1 + t_3) \gamma_2) (-(-1 + t_1) \delta_1 - (-1 + t_2) \delta_2) \&& \\
& (t_2 - t_3) \delta_2 ((-1 + t_2) \delta_1 + (-1 + t_3) \delta_2) + \\
& \quad (-1 + t_1) \gamma_1 ((-1 + t_1) \delta_1 + (-1 + t_3) \delta_2) (1 + (-1 + t_2) \delta_1 + (-1 + t_3) \delta_2) + \\
& \quad (-1 + t_2) \gamma_2 ((-1 + t_1) \delta_1 + (-1 + t_3) \delta_2) (1 + (-1 + t_2) \delta_1 + (-1 + t_3) \delta_2) = 0
\end{aligned}$$

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sols3 = Union[Union /@ SolveAlways[eqns /. reds, {t1, t2, t3}]]
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$$\{\{\gamma_1 \rightarrow 0, \gamma_2 \rightarrow 0, \delta_2 \rightarrow 0\}, \{\gamma_1 \rightarrow 0, \delta_1 \rightarrow 0, \delta_2 \rightarrow 0\}\}$$

```
MatrixForm /@ (U[[1, 2]][1;;2, 1;;2] /. reds /. sols3)
```

$$\left\{ \begin{pmatrix} 1 & \delta_1 - t_1 \delta_1 \\ 0 & 1 - \delta_1 + t_1 \delta_1 \end{pmatrix}, \begin{pmatrix} 1 + \gamma_2 - t_2 \gamma_2 & 0 \\ -\gamma_2 + t_2 \gamma_2 & 1 \end{pmatrix} \right\}$$

```
Inverse[{{1, \delta1 - t1 \delta1}, {0, 1 - \delta1 + t1 \delta1}}] // MatrixForm
```

$$\begin{pmatrix} 1 & \frac{-\delta_1 + t_1 \delta_1}{1 - \delta_1 + t_1 \delta_1} \\ 0 & \frac{1}{1 - \delta_1 + t_1 \delta_1} \end{pmatrix}$$

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Eigenvalues[{{1, \delta1 - t1 \delta1}, {0, 1 - \delta1 + t1 \delta1}}]
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$$\{1, 1 - \delta_1 + t_1 \delta_1\}$$