```
U[i_, j_] := ReplacePart[IdentityMatrix[3], {
    {i, i} → α₀ + α₁ tᵢ + α₂ tⱼ, {i, j} → β₀ + β₁ tᵢ + β₂ tⱼ,
    {j, i} → γ₀ + γ₁ tᵢ + γ₂ tⱼ, {j, j} → δ₀ + δ₁ tᵢ + δ₂ tⱼ
  }]
```

```
U[1, 2].U[1, 3].U[2, 3] // MatrixForm
```

$$\begin{pmatrix} (\alpha_0 + t_1\,\alpha_1 + t_2\,\alpha_2)\ (\alpha_0 + t_1\,\alpha_1 + t_3\,\alpha_2) & (\alpha_0 + t_2\,\alpha_1 + t_3\,\alpha_2)\ (\beta_0 + t_1\,\beta_1 + t_2\,\beta_2) + (\alpha_0 + t_1\,\alpha_1 + t_2\,\alpha_2)\ (\\ (\alpha_0 + t_1\,\alpha_1 + t_3\,\alpha_2)\ (\gamma_0 + t_1\,\gamma_1 + t_2\,\gamma_2) & (\beta_0 + t_1\,\beta_1 + t_3\,\beta_2)\ (\gamma_0 + t_1\,\gamma_1 + t_2\,\gamma_2)\ (\gamma_0 + t_2\,\gamma_1 + t_3\,\gamma_2) + (\\ \gamma_0 + t_1\,\gamma_1 + t_3\,\gamma_2 & (\gamma_0 + t_2\,\gamma_1 + t_3\,\gamma_2)\ (\delta_0 + t_1\,\delta_1 + \end{pmatrix}$$

```
U[2, 3].U[1, 3].U[1, 2] // MatrixForm
```

$$\begin{pmatrix} & (\alpha_0 + t_1\,\alpha_1 + t_2\,\alpha_2)\ (\alpha_0 + t_1\,\alpha_1 + t_3\,\alpha_2) \\ (\alpha_0 + t_2\,\alpha_1 + t_3\,\alpha_2)\ (\gamma_0 + t_1\,\gamma_1 + t_2\,\gamma_2) + (\alpha_0 + t_1\,\alpha_1 + t_2\,\alpha_2)\ (\beta_0 + t_2\,\beta_1 + t_3\,\beta_2)\ (\gamma_0 + t_1\,\gamma_1 + t_3\,\gamma_2) & (\\ (\gamma_0 + t_1\,\gamma_1 + t_2\,\gamma_2)\ (\gamma_0 + t_2\,\gamma_1 + t_3\,\gamma_2) + (\alpha_0 + t_1\,\alpha_1 + t_2\,\alpha_2)\ (\gamma_0 + t_1\,\gamma_1 + t_3\,\gamma_2)\ (\delta_0 + t_2\,\delta_1 + t_3\,\delta_2) & (\end{pmatrix}$$

```
eqns = Simplify[And @@
    Thread[Flatten[U[1, 2].U[1, 3].U[2, 3]] == Flatten[U[2, 3].U[1, 3].U[1, 2]]]]]
```

$(\alpha_0 + t_2\,\alpha_1 + t_3\,\alpha_2)\ (\beta_0 + t_1\,\beta_1 + t_2\,\beta_2) + (\alpha_0 + t_1\,\alpha_1 + t_2\,\alpha_2)\ (\beta_0 + t_1\,\beta_1 + t_3\,\beta_2)\ (\gamma_0 + t_2\,\gamma_1 + t_3\,\gamma_2) ==$
  $(\alpha_0 + t_1\,\alpha_1 + t_3\,\alpha_2)\ (\beta_0 + t_1\,\beta_1 + t_2\,\beta_2) \&\& (\beta_0 + t_1\,\beta_1 + t_2\,\beta_2)\ (\beta_0 + t_2\,\beta_1 + t_3\,\beta_2) +$
  $(\alpha_0 + t_1\,\alpha_1 + t_2\,\alpha_2)\ (\beta_0 + t_1\,\beta_1 + t_3\,\beta_2)\ (\delta_0 + t_2\,\delta_1 + t_3\,\delta_2) == \beta_0 + t_1\,\beta_1 + t_3\,\beta_2 \&\&$
  $(\alpha_0 + t_1\,\alpha_1 + t_3\,\alpha_2)\ (\gamma_0 + t_1\,\gamma_1 + t_2\,\gamma_2) == (\alpha_0 + t_2\,\alpha_1 + t_3\,\alpha_2)\ (\gamma_0 + t_1\,\gamma_1 + t_2\,\gamma_2) +$
  $(\alpha_0 + t_1\,\alpha_1 + t_2\,\alpha_2)\ (\beta_0 + t_2\,\beta_1 + t_3\,\beta_2)\ (\gamma_0 + t_1\,\gamma_1 + t_3\,\gamma_2) \&\&$
  $(\beta_0 + t_1\,\beta_1 + t_3\,\beta_2)\ (\gamma_0 + t_1\,\gamma_1 + t_2\,\gamma_2)\ (\gamma_0 + t_2\,\gamma_1 + t_3\,\gamma_2) ==$
  $(\beta_0 + t_1\,\beta_1 + t_2\,\beta_2)\ (\beta_0 + t_2\,\beta_1 + t_3\,\beta_2)\ (\gamma_0 + t_1\,\gamma_1 + t_3\,\gamma_2) \&\&$
  $(\beta_0 + t_2\,\beta_1 + t_3\,\beta_2)\ (\delta_0 + t_1\,\delta_1 + t_2\,\delta_2) + (\beta_0 + t_1\,\beta_1 + t_3\,\beta_2)\ (\gamma_0 + t_1\,\gamma_1 + t_2\,\gamma_2)\ (\delta_0 + t_2\,\delta_1 + t_3\,\delta_2) ==$
  $(\beta_0 + t_2\,\beta_1 + t_3\,\beta_2)\ (\delta_0 + t_1\,\delta_1 + t_3\,\delta_2) \&\&$
  $\gamma_0 + t_1\,\gamma_1 + t_3\,\gamma_2 == (\gamma_0 + t_1\,\gamma_1 + t_2\,\gamma_2)\ (\gamma_0 + t_2\,\gamma_1 + t_3\,\gamma_2) +$
  $(\alpha_0 + t_1\,\alpha_1 + t_2\,\alpha_2)\ (\gamma_0 + t_1\,\gamma_1 + t_3\,\gamma_2)\ (\delta_0 + t_2\,\delta_1 + t_3\,\delta_2) \&\&$
  $(\gamma_0 + t_2\,\gamma_1 + t_3\,\gamma_2)\ (\delta_0 + t_1\,\delta_1 + t_3\,\delta_2) == (\gamma_0 + t_2\,\gamma_1 + t_3\,\gamma_2)\ (\delta_0 + t_1\,\delta_1 + t_2\,\delta_2) +$
  $(\beta_0 + t_1\,\beta_1 + t_2\,\beta_2)\ (\gamma_0 + t_1\,\gamma_1 + t_3\,\gamma_2)\ (\delta_0 + t_2\,\delta_1 + t_3\,\delta_2)$

```
sols0 = Union[Union /@ SolveAlways[eqns, {t₁, t₂, t₃}]]
```

$Aborted

```
sols1 = Union[Union /@ SolveAlways[eqns && (α₀ + α₁ + α₂ == 1) &&
        (β₀ + β₁ + β₂ == 0) && (γ₀ + γ₁ + γ₂ == 0) && (δ₀ + δ₁ + δ₂ == 1), {t₁, t₂, t₃}]]
```

$Aborted

```
sols2 = Union[
  Union /@ SolveAlways[eqns && (α₀ + α₁ + α₂ == 1) && (β₀ + β₁ + β₂ == 0) && (γ₀ + γ₁ + γ₂ == 0) &&
        (δ₀ + δ₁ + δ₂ == 1) && (α₀ + γ₀ == 1) && (α₁ + γ₁ == 0) && (α₂ + γ₂ == 0) &&
        (β₀ + δ₀ == 1) && (β₁ + δ₁ == 0) && (β₂ + δ₂ == 0), {t₁, t₂, t₃}]]
```

$Aborted

```
reds = Solve[(α₀ + α₁ + α₂ == 1) && (β₀ + β₁ + β₂ == 0) &&
        (γ₀ + γ₁ + γ₂ == 0) && (δ₀ + δ₁ + δ₂ == 1) && (α₀ + γ₀ == 1) && (α₁ + γ₁ == 0) &&
        (α₂ + γ₂ == 0) && (β₀ + δ₀ == 1) && (β₁ + δ₁ == 0) && (β₂ + δ₂ == 0)] // First
```

$\{\alpha_0 \to 1 + \gamma_1 + \gamma_2,\ \alpha_1 \to -\gamma_1,\ \alpha_2 \to -\gamma_2,$
$\ \beta_0 \to \delta_1 + \delta_2,\ \beta_1 \to -\delta_1,\ \beta_2 \to -\delta_2,\ \gamma_0 \to -\gamma_1 - \gamma_2,\ \delta_0 \to 1 - \delta_1 - \delta_2\}$

Simplify[eqns /. reds]

$((-1 + t_1) \gamma_1 + (-1 + t_2) \gamma_2) (1 - (-1 + t_1) \gamma_1 - (-1 + t_3) \gamma_2) ==$
$\quad ((-1 + t_1) \gamma_1 + (-1 + t_2) \gamma_2) (1 - (-1 + t_2) \gamma_1 - (-1 + t_3) \gamma_2) +$
$\quad (1 - (-1 + t_1) \gamma_1 - (-1 + t_2) \gamma_2) ((-1 + t_1) \gamma_1 + (-1 + t_3) \gamma_2) (- (-1 + t_2) \delta_1 - (-1 + t_3) \delta_2) \&\&$
$\quad (-1 + t_1) \gamma_1 + (-1 + t_3) \gamma_2 == ((-1 + t_1) \gamma_1 + (-1 + t_2) \gamma_2) ((-1 + t_2) \gamma_1 + (-1 + t_3) \gamma_2) +$
$\quad (1 - (-1 + t_1) \gamma_1 - (-1 + t_2) \gamma_2)$
$\quad\quad ((-1 + t_1) \gamma_1 + (-1 + t_3) \gamma_2) (1 + (-1 + t_2) \delta_1 + (-1 + t_3) \delta_2) \&\&$
$\quad ((-1 + t_1) \gamma_1 + (-1 + t_2) \gamma_2) ((-1 + t_2) \gamma_1 + (-1 + t_3) \gamma_2) (- (-1 + t_1) \delta_1 - (-1 + t_3) \delta_2) ==$
$\quad ((-1 + t_1) \gamma_1 + (-1 + t_3) \gamma_2) (- (-1 + t_1) \delta_1 - (-1 + t_2) \delta_2) (- (-1 + t_2) \delta_1 - (-1 + t_3) \delta_2) \&\&$
$\quad ((-1 + t_2) \gamma_1 + (-1 + t_3) \gamma_2) (1 + (-1 + t_1) \delta_1 + (-1 + t_3) \delta_2) ==$
$\quad ((-1 + t_2) \gamma_1 + (-1 + t_3) \gamma_2) (1 + (-1 + t_1) \delta_1 + (-1 + t_2) \delta_2) +$
$\quad\quad ((-1 + t_1) \gamma_1 + (-1 + t_3) \gamma_2) (- (-1 + t_1) \delta_1 - (-1 + t_2) \delta_2) (1 + (-1 + t_2) \delta_1 + (-1 + t_3) \delta_2) \&\&$
$\quad (1 - (-1 + t_2) \gamma_1 - (-1 + t_3) \gamma_2) (- (-1 + t_1) \delta_1 - (-1 + t_2) \delta_2) +$
$\quad\quad (1 - (-1 + t_1) \gamma_1 - (-1 + t_2) \gamma_2) ((-1 + t_2) \gamma_1 + (-1 + t_3) \gamma_2) (- (-1 + t_1) \delta_1 - (-1 + t_3) \delta_2) ==$
$\quad (1 - (-1 + t_1) \gamma_1 - (-1 + t_3) \gamma_2) (- (-1 + t_1) \delta_1 - (-1 + t_2) \delta_2) \&\&$
$\quad (t_2 - t_3) \delta_2 ((-1 + t_2) \delta_1 + (-1 + t_3) \delta_2) +$
$\quad\quad (-1 + t_1) \gamma_1 ((-1 + t_1) \delta_1 + (-1 + t_3) \delta_2) (1 + (-1 + t_2) \delta_1 + (-1 + t_3) \delta_2) +$
$\quad\quad (-1 + t_2) \gamma_2 ((-1 + t_1) \delta_1 + (-1 + t_3) \delta_2) (1 + (-1 + t_2) \delta_1 + (-1 + t_3) \delta_2) == 0$

sols3 = Union[Union /@ SolveAlways[eqns /. reds, {t₁, t₂, t₃}]]

$\{\{\gamma_1 \to 0, \gamma_2 \to 0, \delta_2 \to 0\}, \{\gamma_1 \to 0, \delta_1 \to 0, \delta_2 \to 0\}\}$

MatrixForm /@ (U[1, 2]⟦1 ;; 2, 1 ;; 2⟧ /. reds /. sols3)

$$\left\{ \begin{pmatrix} 1 & \delta_1 - t_1 \delta_1 \\ 0 & 1 - \delta_1 + t_1 \delta_1 \end{pmatrix}, \begin{pmatrix} 1 + \gamma_2 - t_2 \gamma_2 & 0 \\ -\gamma_2 + t_2 \gamma_2 & 1 \end{pmatrix} \right\}$$

Inverse$\left[\begin{pmatrix} 1 & \delta_1 - t_1 \delta_1 \\ 0 & 1 - \delta_1 + t_1 \delta_1 \end{pmatrix} \right]$ // MatrixForm

$$\begin{pmatrix} 1 & \dfrac{-\delta_1 + t_1 \delta_1}{1 - \delta_1 + t_1 \delta_1} \\ 0 & \dfrac{1}{1 - \delta_1 + t_1 \delta_1} \end{pmatrix}$$

Eigenvalues$\left[\begin{pmatrix} 1 & \delta_1 - t_1 \delta_1 \\ 0 & 1 - \delta_1 + t_1 \delta_1 \end{pmatrix} \right]$

$\{1, 1 - \delta_1 + t_1 \delta_1\}$