$$\mathbf{U1}[x\_,\ y\_] := \begin{pmatrix}\alpha_{x,y} & \beta_{x,y} & 0\\ \gamma_{x,y} & \delta_{x,y} & 0\\ 0 & 0 & 1\end{pmatrix};\ \mathbf{U2}[x\_,\ y\_] := \begin{pmatrix}1 & 0 & 0\\ 0 & \alpha_{x,y} & \beta_{x,y}\\ 0 & \gamma_{x,y} & \delta_{x,y}\end{pmatrix};$$

```
U1[1, 2].U2[1, 3].U1[2, 3] // MatrixForm
```

$$\begin{pmatrix}\alpha_{1,2}\,\alpha_{2,3}+\alpha_{1,3}\,\beta_{1,2}\,\gamma_{2,3} & \alpha_{1,2}\,\beta_{2,3}+\alpha_{1,3}\,\beta_{1,2}\,\delta_{2,3} & \beta_{1,2}\,\beta_{1,3}\\ \alpha_{2,3}\,\gamma_{1,2}+\alpha_{1,3}\,\gamma_{2,3}\,\delta_{1,2} & \beta_{2,3}\,\gamma_{1,2}+\alpha_{1,3}\,\delta_{1,2}\,\delta_{2,3} & \beta_{1,3}\,\delta_{1,2}\\ \gamma_{1,3}\,\gamma_{2,3} & \gamma_{1,3}\,\delta_{2,3} & \delta_{1,3}\end{pmatrix}$$

```
U2[2, 3].U1[1, 3].U2[1, 2] // MatrixForm
```

$$\begin{pmatrix}\alpha_{1,3} & \alpha_{1,2}\,\beta_{1,3} & \beta_{1,2}\,\beta_{1,3}\\ \alpha_{2,3}\,\gamma_{1,3} & \beta_{2,3}\,\gamma_{1,2}+\alpha_{1,2}\,\alpha_{2,3}\,\delta_{1,3} & \beta_{2,3}\,\delta_{1,2}+\alpha_{2,3}\,\beta_{1,2}\,\delta_{1,3}\\ \gamma_{1,3}\,\gamma_{2,3} & \alpha_{1,2}\,\gamma_{2,3}\,\delta_{1,3}+\gamma_{1,2}\,\delta_{2,3} & \beta_{1,2}\,\gamma_{2,3}\,\delta_{1,3}+\delta_{1,2}\,\delta_{2,3}\end{pmatrix}$$

```
eqns = Simplify[And @@ Thread[
    Flatten[U1[1, 2].U2[1, 3].U1[2, 3]] == Flatten[U2[2, 3].U1[1, 3].U2[1, 2]]]]
```

$$\begin{aligned}&\alpha_{1,2}\,\alpha_{2,3}+\alpha_{1,3}\,\beta_{1,2}\,\gamma_{2,3}==\alpha_{1,3}\ \&\&\\ &\alpha_{1,2}\,(\beta_{1,3}-\beta_{2,3})==\alpha_{1,3}\,\beta_{1,2}\,\delta_{2,3}\ \&\&\ \alpha_{2,3}\,(\gamma_{1,2}-\gamma_{1,3})+\alpha_{1,3}\,\gamma_{2,3}\,\delta_{1,2}==0\ \&\&\\ &\alpha_{1,2}\,\alpha_{2,3}\,\delta_{1,3}==\alpha_{1,3}\,\delta_{1,2}\,\delta_{2,3}\ \&\&\ \beta_{1,3}\,\delta_{1,2}==\beta_{2,3}\,\delta_{1,2}+\alpha_{2,3}\,\beta_{1,2}\,\delta_{1,3}\ \&\&\\ &\alpha_{1,2}\,\gamma_{2,3}\,\delta_{1,3}+(\gamma_{1,2}-\gamma_{1,3})\,\delta_{2,3}==0\ \&\&\ \delta_{1,3}==\beta_{1,2}\,\gamma_{2,3}\,\delta_{1,3}+\delta_{1,2}\,\delta_{2,3}\end{aligned}$$

$$\mathbf{eqns1} = \mathbf{Simplify}\left[\mathbf{eqns}\ /.\ \xi_{i\_,j\_} :\to \xi_0+\xi_1\,t_i+\xi_2\,t_j\right]$$

$$\begin{aligned}&(\alpha_0+t_1\alpha_1+t_2\alpha_2)\,(\alpha_0+t_2\alpha_1+t_3\alpha_2)+(\alpha_0+t_1\alpha_1+t_3\alpha_2)\,(\beta_0+t_1\beta_1+t_2\beta_2)\,(\gamma_0+t_2\gamma_1+t_3\gamma_2)==\\ &\quad \alpha_0+t_1\alpha_1+t_3\alpha_2\ \&\&\ (t_1-t_2)\,(\alpha_0+t_1\alpha_1+t_2\alpha_2)\,\beta_1==\\ &\quad (\alpha_0+t_1\alpha_1+t_3\alpha_2)\,(\beta_0+t_1\beta_1+t_2\beta_2)\,(\delta_0+t_2\delta_1+t_3\delta_2)\ \&\&\\ &(t_2-t_3)\,(\alpha_0+t_2\alpha_1+t_3\alpha_2)\,\gamma_2+(\alpha_0+t_1\alpha_1+t_3\alpha_2)\,(\gamma_0+t_2\gamma_1+t_3\gamma_2)\,(\delta_0+t_1\delta_1+t_2\delta_2)==0\ \&\&\\ &(\alpha_0+t_1\alpha_1+t_2\alpha_2)\,(\alpha_0+t_2\alpha_1+t_3\alpha_2)\,(\delta_0+t_1\delta_1+t_3\delta_2)==\\ &\quad (\alpha_0+t_1\alpha_1+t_3\alpha_2)\,(\delta_0+t_1\delta_1+t_2\delta_2)\,(\delta_0+t_2\delta_1+t_3\delta_2)\ \&\&\\ &(\beta_0+t_1\beta_1+t_3\beta_2)\,(\delta_0+t_1\delta_1+t_2\delta_2)==(\beta_0+t_2\beta_1+t_3\beta_2)\,(\delta_0+t_1\delta_1+t_2\delta_2)+\\ &\quad (\alpha_0+t_2\alpha_1+t_3\alpha_2)\,(\beta_0+t_1\beta_1+t_2\beta_2)\,(\delta_0+t_1\delta_1+t_3\delta_2)\ \&\&\\ &(\alpha_0+t_1\alpha_1+t_2\alpha_2)\,(\gamma_0+t_2\gamma_1+t_3\gamma_2)\,(\delta_0+t_1\delta_1+t_3\delta_2)+(t_2-t_3)\,\gamma_2\,(\delta_0+t_2\delta_1+t_3\delta_2)==0\ \&\&\\ &\delta_0+t_1\delta_1+t_3\delta_2==\\ &\quad (\beta_0+t_1\beta_1+t_2\beta_2)\,(\gamma_0+t_2\gamma_1+t_3\gamma_2)\,(\delta_0+t_1\delta_1+t_3\delta_2)+(\delta_0+t_1\delta_1+t_2\delta_2)\,(\delta_0+t_2\delta_1+t_3\delta_2)\end{aligned}$$

$$\mathbf{sols0} = \mathbf{Union}[\mathbf{Union}\ /@\ \mathbf{SolveAlways}[\mathbf{eqns1},\ \{t_1,\ t_2,\ t_3\}]]$$

```
$Aborted
```

$$\begin{aligned}&\mathbf{sols} =\\ &\ \mathbf{Union}[\mathbf{Union}\ /@\ \mathbf{SolveAlways}[\mathbf{eqns1}\ /.\ \{\alpha_0\to 0,\ \beta_0\to 1,\ \gamma_0\to 1,\ \delta_0\to 0\},\ \{t_1,\ t_2,\ t_3\}]]\end{aligned}$$

$$\begin{aligned}&\{\{\alpha_1\to 0,\ \alpha_2\to 0,\ \delta_1\to 0,\ \delta_2\to 0\},\ \{\alpha_1\to 0,\ \alpha_2\to 0,\ \beta_1\to 0,\ \beta_2\to 0,\ \gamma_1\to -\delta_1,\ \gamma_2\to 0,\ \delta_2\to 0\},\\ &\ \{\alpha_1\to 0,\ \alpha_2\to 0,\ \beta_1\to 0,\ \beta_2\to 0,\ \gamma_1\to -\delta_2,\ \gamma_2\to 0,\ \delta_1\to 0\},\\ &\ \{\alpha_1\to 0,\ \alpha_2\to 0,\ \beta_1\to 0,\ \beta_2\to -\delta_1,\ \gamma_1\to 0,\ \gamma_2\to 0,\ \delta_2\to 0\},\\ &\ \{\alpha_1\to 0,\ \alpha_2\to 0,\ \beta_1\to 0,\ \beta_2\to -\delta_2,\ \gamma_1\to 0,\ \gamma_2\to 0,\ \delta_1\to 0\},\\ &\ \{\alpha_1\to 0,\ \beta_1\to 0,\ \beta_2\to 0,\ \gamma_1\to -\alpha_2,\ \gamma_2\to 0,\ \delta_1\to 0,\ \delta_2\to 0\},\\ &\ \{\alpha_1\to 0,\ \beta_1\to 0,\ \beta_2\to -\alpha_2,\ \gamma_1\to 0,\ \gamma_2\to 0,\ \delta_1\to 0,\ \delta_2\to 0\},\\ &\ \{\alpha_2\to 0,\ \beta_1\to 0,\ \beta_2\to 0,\ \gamma_1\to -\alpha_1,\ \gamma_2\to 0,\ \delta_1\to 0,\ \delta_2\to 0\},\\ &\ \{\alpha_2\to 0,\ \beta_1\to 0,\ \beta_2\to -\alpha_1,\ \gamma_1\to 0,\ \gamma_2\to 0,\ \delta_1\to 0,\ \delta_2\to 0\}\}\end{aligned}$$

MatrixForm[

$$\begin{pmatrix} \alpha_{x,y} & \beta_{x,y} \\ \gamma_{x,y} & \delta_{x,y} \end{pmatrix} \;/.\; \xi_{x\_,y\_} \mapsto \xi_0 + \xi_1\, x + \xi_2\, y \;/.\; \{\alpha_0 \to 0,\ \beta_0 \to 1,\ \gamma_0 \to 1,\ \delta_0 \to 0\}\Big] \;/.\; \text{sols}$$

$$\left\{\begin{pmatrix} 0 & 1 + x\,\beta_1 + y\,\beta_2 \\ 1 + x\,\gamma_1 + y\,\gamma_2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 - x\,\delta_1 & x\,\delta_1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 - x\,\delta_2 & y\,\delta_2 \end{pmatrix}, \begin{pmatrix} 0 & 1 - y\,\delta_1 \\ 1 & x\,\delta_1 \end{pmatrix},\right.$$
$$\left.\begin{pmatrix} 0 & 1 - y\,\delta_2 \\ 1 & y\,\delta_2 \end{pmatrix}, \begin{pmatrix} y\,\alpha_2 & 1 \\ 1 - x\,\alpha_2 & 0 \end{pmatrix}, \begin{pmatrix} y\,\alpha_2 & 1 - y\,\alpha_2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} x\,\alpha_1 & 1 \\ 1 - x\,\alpha_1 & 0 \end{pmatrix}, \begin{pmatrix} x\,\alpha_1 & 1 - y\,\alpha_1 \\ 1 & 0 \end{pmatrix}\right\}$$

Eigenvalues /@

$$\left(\begin{pmatrix} \alpha_{x,y} & \beta_{x,y} \\ \gamma_{x,y} & \delta_{x,y} \end{pmatrix} \;/.\; \xi_{x\_,y\_} \mapsto \xi_0 + \xi_1\, x + \xi_2\, y \;/.\; \{\alpha_0 \to 0,\ \beta_0 \to 1,\ \gamma_0 \to 1,\ \delta_0 \to 0\} \;/.\; \text{sols}\right)$$

$$\Big\{\Big\{-\sqrt{1 + x\,\beta_1 + y\,\beta_2}\,\sqrt{1 + x\,\gamma_1 + y\,\gamma_2},\ \sqrt{1 + x\,\beta_1 + y\,\beta_2}\,\sqrt{1 + x\,\gamma_1 + y\,\gamma_2}\Big\},$$
$$\{1, -1 + x\,\delta_1\},\ \Big\{\frac{1}{2}\Big(y\,\delta_2 - \sqrt{4 - 4\,x\,\delta_2 + y^2\,\delta_2^2}\Big),\ \frac{1}{2}\Big(y\,\delta_2 + \sqrt{4 - 4\,x\,\delta_2 + y^2\,\delta_2^2}\Big)\Big\},$$
$$\Big\{\frac{1}{2}\Big(x\,\delta_1 - \sqrt{4 - 4\,y\,\delta_1 + x^2\,\delta_1^2}\Big),\ \frac{1}{2}\Big(x\,\delta_1 + \sqrt{4 - 4\,y\,\delta_1 + x^2\,\delta_1^2}\Big)\Big\},\ \{1, -1 + y\,\delta_2\},$$
$$\Big\{\frac{1}{2}\Big(y\,\alpha_2 - \sqrt{4 - 4\,x\,\alpha_2 + y^2\,\alpha_2^2}\Big),\ \frac{1}{2}\Big(y\,\alpha_2 + \sqrt{4 - 4\,x\,\alpha_2 + y^2\,\alpha_2^2}\Big)\Big\},\ \{1, -1 + y\,\alpha_2\},$$
$$\{1, -1 + x\,\alpha_1\},\ \Big\{\frac{1}{2}\Big(x\,\alpha_1 - \sqrt{4 - 4\,y\,\alpha_1 + x^2\,\alpha_1^2}\Big),\ \frac{1}{2}\Big(x\,\alpha_1 + \sqrt{4 - 4\,y\,\alpha_1 + x^2\,\alpha_1^2}\Big)\Big\}\Big\}$$

eqns2 = **Simplify**[**eqns** /. $\xi_{i\_,j\_} \mapsto \xi_0 + \xi_1\,(t_i - 1) + \xi_2\,(t_j - 1)$]

$$(\alpha_0 + (-1 + t_1)\,\alpha_1 + (-1 + t_2)\,\alpha_2)\,(\alpha_0 + (-1 + t_2)\,\alpha_1 + (-1 + t_3)\,\alpha_2) +$$
$$(\alpha_0 + (-1 + t_1)\,\alpha_1 + (-1 + t_3)\,\alpha_2)\,(\beta_0 + (-1 + t_1)\,\beta_1 + (-1 + t_2)\,\beta_2)$$
$$(\gamma_0 + (-1 + t_2)\,\gamma_1 + (-1 + t_3)\,\gamma_2) == \alpha_0 + (-1 + t_1)\,\alpha_1 + (-1 + t_3)\,\alpha_2\ \&\&$$
$$(t_1 - t_2)\,(\alpha_0 + (-1 + t_1)\,\alpha_1 + (-1 + t_2)\,\alpha_2)\,\beta_1 == (\alpha_0 + (-1 + t_1)\,\alpha_1 + (-1 + t_3)\,\alpha_2)$$
$$(\beta_0 + (-1 + t_1)\,\beta_1 + (-1 + t_2)\,\beta_2)\,(\delta_0 + (-1 + t_2)\,\delta_1 + (-1 + t_3)\,\delta_2)\ \&\&$$
$$(t_2 - t_3)\,(\alpha_0 + (-1 + t_2)\,\alpha_1 + (-1 + t_3)\,\alpha_2)\,\gamma_2 + (\alpha_0 + (-1 + t_1)\,\alpha_1 + (-1 + t_3)\,\alpha_2)$$
$$(\gamma_0 + (-1 + t_2)\,\gamma_1 + (-1 + t_3)\,\gamma_2)\,(\delta_0 + (-1 + t_1)\,\delta_1 + (-1 + t_2)\,\delta_2) == 0\ \&\&$$
$$(\alpha_0 + (-1 + t_1)\,\alpha_1 + (-1 + t_2)\,\alpha_2)\,(\alpha_0 + (-1 + t_2)\,\alpha_1 + (-1 + t_3)\,\alpha_2)$$
$$(\delta_0 + (-1 + t_1)\,\delta_1 + (-1 + t_3)\,\delta_2) == (\alpha_0 + (-1 + t_1)\,\alpha_1 + (-1 + t_3)\,\alpha_2)$$
$$(\delta_0 + (-1 + t_1)\,\delta_1 + (-1 + t_2)\,\delta_2)\,(\delta_0 + (-1 + t_2)\,\delta_1 + (-1 + t_3)\,\delta_2)\ \&\&$$
$$(\beta_0 + (-1 + t_1)\,\beta_1 + (-1 + t_3)\,\beta_2)\,(\delta_0 + (-1 + t_1)\,\delta_1 + (-1 + t_2)\,\delta_2) ==$$
$$(\beta_0 + (-1 + t_2)\,\beta_1 + (-1 + t_3)\,\beta_2)\,(\delta_0 + (-1 + t_1)\,\delta_1 + (-1 + t_2)\,\delta_2) +$$
$$(\alpha_0 + (-1 + t_2)\,\alpha_1 + (-1 + t_3)\,\alpha_2)$$
$$(\beta_0 + (-1 + t_1)\,\beta_1 + (-1 + t_2)\,\beta_2)\,(\delta_0 + (-1 + t_1)\,\delta_1 + (-1 + t_3)\,\delta_2)\ \&\&$$
$$(\alpha_0 + (-1 + t_1)\,\alpha_1 + (-1 + t_2)\,\alpha_2)\,(\gamma_0 + (-1 + t_2)\,\gamma_1 + (-1 + t_3)\,\gamma_2)$$
$$(\delta_0 + (-1 + t_1)\,\delta_1 + (-1 + t_3)\,\delta_2) + (t_2 - t_3)\,\gamma_2\,(\delta_0 + (-1 + t_2)\,\delta_1 + (-1 + t_3)\,\delta_2) == 0\ \&\&$$
$$\delta_0 + (-1 + t_1)\,\delta_1 + (-1 + t_3)\,\delta_2 == (\beta_0 + (-1 + t_1)\,\beta_1 + (-1 + t_2)\,\beta_2)$$
$$(\gamma_0 + (-1 + t_2)\,\gamma_1 + (-1 + t_3)\,\gamma_2)\,(\delta_0 + (-1 + t_1)\,\delta_1 + (-1 + t_3)\,\delta_2) +$$
$$(\delta_0 + (-1 + t_1)\,\delta_1 + (-1 + t_2)\,\delta_2)\,(\delta_0 + (-1 + t_2)\,\delta_1 + (-1 + t_3)\,\delta_2)$$

sols2 =
Union[**Union** /@ **SolveAlways**[**eqns2** /. $\{\alpha_0 \to 0,\ \beta_0 \to 1,\ \gamma_0 \to 1,\ \delta_0 \to 0\}$, $\{t_1, t_2, t_3\}$]]

$Aborted

```
MatrixForm[ (α_{x,y}  β_{x,y}
             γ_{x,y}  δ_{x,y}) /. ζ_{_x_,y_} :→ ζ_0 + ζ_1 (x - 1) + ζ_2 (y - 1) /.
    {α_0 → 0, β_0 → 1, γ_0 → 1, δ_0 → 0}] /. sols2

Eigenvalues /@ ((α_{x,y}  β_{x,y}
                 γ_{x,y}  δ_{x,y}) /. ζ_{_x_,y_} :→ ζ_0 + ζ_1 (x - 1) + ζ_2 (y - 1) /.
    {α_0 → 0, β_0 → 1, γ_0 → 1, δ_0 → 0} /. sols2)
```