Pensieve header: Demos for my Bannoye-1407 talk.
The .nb (source) file is at http://drorbn.net/AcademicPensieve/2014-07/GoodFormulas/.

```
SetDirectory["C:/drorbn/AcademicPensieve/2014-07/GoodFormulas/"]
```

```
C:\drorbn\AcademicPensieve\2014-07\GoodFormulas
```

```
<< KnotTheory`
```

```
Loading KnotTheory` version of April 3, 2014, 16:23:56.0784.
Read more at http://katlas.org/wiki/KnotTheory.
```

Initialization

```
ΓCollect[Γ[ω_, λ_]] := Γ[Factor[ω],
   Collect[λ, h_, Collect[#, t_, Factor] &]];
Format[Γ[ω_, λ_]] := Module[{S, M},
   S = Union@Cases[Γ[ω, λ], (h | t)_a_ :> a, ∞];
   M = Outer[Factor[∂_{h_#1 t_#2} λ] &, S, S];
   M = Prepend[M, t_# & /@ S] // Transpose;
   M = Prepend[M, Prepend[h_# & /@ S, ω]];
   M // MatrixForm];
```

Program

```
Γ /: Γ[ω1_, λ1_] Γ[ω2_, λ2_] := Γ[ω1 * ω2, λ1 + λ2];
m_{a_ b_ → c_}[Γ[ω_, λ_]] := Module[{α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ, μ},
   {{α, β, θ}, {γ, δ, ϵ}, {ϕ, ψ, Ξ}} =
     {{∂_{t_a,h_a} λ, ∂_{t_a,h_b} λ, ∂_{t_a} λ},
      {∂_{t_b,h_a} λ, ∂_{t_b,h_b} λ, ∂_{t_b} λ},
      {∂_{h_a} λ, ∂_{h_b} λ, λ}} /. (t | h)_{a|b} → 0;
   Γ[(μ = 1 - β) ω, {t_c, 1}.{{γ + α δ / μ, ϵ + δ θ / μ},
        {ϕ + α ψ / μ, Ξ + ψ θ / μ}}.{h_c, 1}]
     /. {T_a → T_c, T_b → T_c} // ΓCollect];
Rp_{a_ b_} := Γ[1, {t_a, t_b}.{{1, 1 - T_a}, {0, T_a}}.{h_a, h_b}];
Rm_{a_ b_} := Rp_{ab} /. T_a → 1 / T_a;
```

```
γ = Γ[ω, (t_1 t_2 t_3 t_S).{{α_11, α_12, α_13, θ_1}, {α_21, α_22, α_23, θ_2},
       {α_31, α_32, α_33, θ_3}, {ϕ_1, ϕ_2, ϕ_3, Ξ}}.{h_1, h_2, h_3, h_S} // Total // Total]
```

$$\begin{pmatrix} \omega & h_1 & h_2 & h_3 & h_S \\ t_1 & \alpha_{11} & \alpha_{12} & \alpha_{13} & \theta_1 \\ t_2 & \alpha_{21} & \alpha_{22} & \alpha_{23} & \theta_2 \\ t_3 & \alpha_{31} & \alpha_{32} & \alpha_{33} & \theta_3 \\ t_S & \phi_1 & \phi_2 & \phi_3 & \Xi \end{pmatrix}$$

```
γ // m_{12→1} // m_{13→1}
```

$$\begin{pmatrix} \omega\,(1-\alpha_{12}-\alpha_{13}\,\alpha_{22}-\alpha_{23}+\alpha_{12}\,\alpha_{23}) & h_1 \\ t_1 & \frac{\alpha_{31}-\alpha_{12}\,\alpha_{31}-\alpha_{13}\,\alpha_{22}\,\alpha_{31}-\alpha_{23}\,\alpha_{31}+\alpha_{12}\,\alpha_{23}\,\alpha_{31}+\alpha_{11}\,\alpha_{32}+\alpha_{13}\,\alpha_{21}\,\alpha_{32}-\alpha_{11}\,\alpha_{23}\,\alpha_{32}+\alpha_{21}\,\alpha_{33}-\alpha_{12}\,\alpha_{21}\,\alpha_{33}+}{1-\alpha_{12}-\alpha_{13}\,\alpha_{22}-\alpha_{23}+\alpha_{12}\,\alpha_{23}} \\ t_S & \frac{\phi_1-\alpha_{12}\,\phi_1-\alpha_{13}\,\alpha_{22}\,\phi_1-\alpha_{23}\,\phi_1+\alpha_{12}\,\alpha_{23}\,\phi_1+\alpha_{11}\,\phi_2+\alpha_{13}\,\alpha_{21}\,\phi_2-\alpha_{11}\,\alpha_{23}\,\phi_2+\alpha_{21}\,\phi_3-\alpha_{12}\,\alpha_{21}\,\phi_3+\alpha_{11}}{1-\alpha_{12}-\alpha_{13}\,\alpha_{22}-\alpha_{23}+\alpha_{12}\,\alpha_{23}} \end{pmatrix}$$

γ // $m_{23\to 2}$ // $m_{12\to 1}$

$$\begin{pmatrix} \omega\,(1-\alpha_{12}-\alpha_{13}\,\alpha_{22}-\alpha_{23}+\alpha_{12}\,\alpha_{23}) & h_1 \\ t_1 & \frac{\alpha_{31}-\alpha_{12}\,\alpha_{31}-\alpha_{13}\,\alpha_{22}\,\alpha_{31}-\alpha_{23}\,\alpha_{31}+\alpha_{12}\,\alpha_{23}\,\alpha_{31}+\alpha_{11}\,\alpha_{32}+\alpha_{13}\,\alpha_{21}\,\alpha_{32}-\alpha_{11}\,\alpha_{23}\,\alpha_{32}+\alpha_{21}\,\alpha_{33}-\alpha_{12}\,\alpha_{21}\,\alpha_{33}+}{1-\alpha_{12}-\alpha_{13}\,\alpha_{22}-\alpha_{23}+\alpha_{12}\,\alpha_{23}} \\ t_S & \frac{\phi_1-\alpha_{12}\,\phi_1-\alpha_{13}\,\alpha_{22}\,\phi_1-\alpha_{23}\,\phi_1+\alpha_{12}\,\alpha_{23}\,\phi_1+\alpha_{11}\,\phi_2+\alpha_{13}\,\alpha_{21}\,\phi_2-\alpha_{11}\,\alpha_{23}\,\phi_2+\alpha_{21}\,\phi_3-\alpha_{12}\,\alpha_{21}\,\phi_3+\alpha_{11}}{1-\alpha_{12}-\alpha_{13}\,\alpha_{22}-\alpha_{23}+\alpha_{12}\,\alpha_{23}} \end{pmatrix}$$

MetaAssoc

$$\gamma = \Gamma\left[\omega,\ \{t_1, t_2, t_3, t_S\}.\begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \theta_1 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \theta_2 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \theta_3 \\ \phi_1 & \phi_2 & \phi_3 & \Xi \end{pmatrix}.\{h_1, h_2, h_3, h_S\}\right];$$

(γ // $m_{12\to 1}$ // $m_{13\to 1}$) == (γ // $m_{23\to 2}$ // $m_{12\to 1}$)

MetaAssoc

True

R3

{Rm_{51} Rm_{62} Rp_{34} // $m_{14\to 1}$ // $m_{25\to 2}$ // $m_{36\to 3}$,
Rp_{61} Rm_{24} Rm_{35} // $m_{14\to 1}$ // $m_{25\to 2}$ // $m_{36\to 3}$}

R3

$$\left\{\begin{pmatrix} 1 & h_1 & h_2 & h_3 \\ t_1 & \frac{T_3}{T_2} & 0 & 0 \\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0 \\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1 \end{pmatrix}, \begin{pmatrix} 1 & h_1 & h_2 & h_3 \\ t_1 & \frac{T_3}{T_2} & 0 & 0 \\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0 \\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1 \end{pmatrix}\right\}$$

γ = $Rm_{12,1}$ Rm_{27} Rm_{83} $Rm_{4,11}$ $Rp_{16,5}$ $Rp_{6,13}$ $Rp_{14,9}$ $Rp_{10,15}$

1	h_1	h_2	h_3	h_4	h_5	h_6	h_7	h_8	h_9	h_{10}	h_{11}	h_{12}	h_{13}	h_{14}	h_{15}	h_{16}
t_1	$\frac{1}{T_{12}}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
t_2	0	1	0	0	0	0	$\frac{-1+T_2}{T_2}$	0	0	0	0	0	0	0	0	0
t_3	0	0	$\frac{1}{T_8}$	0	0	0	0	0	0	0	0	0	0	0	0	0
t_4	0	0	0	1	0	0	0	0	0	0	$\frac{-1+T_4}{T_4}$	0	0	0	0	0
t_5	0	0	0	0	T_{16}	0	0	0	0	0	0	0	0	0	0	0
t_6	0	0	0	0	0	1	0	0	0	0	0	0	$1-T_6$	0	0	0
t_7	0	0	0	0	0	0	$\frac{1}{T_2}$	0	0	0	0	0	0	0	0	0
t_8	0	0	$\frac{-1+T_8}{T_8}$	0	0	0	0	1	0	0	0	0	0	0	0	0
t_9	0	0	0	0	0	0	0	0	T_{14}	0	0	0	0	0	0	0
t_{10}	0	0	0	0	0	0	0	0	0	1	0	0	0	0	$1-T_{10}$	0
t_{11}	0	0	0	0	0	0	0	0	0	0	$\frac{1}{T_4}$	0	0	0	0	0
t_{12}	$\frac{-1+T_{12}}{T_{12}}$	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
t_{13}	0	0	0	0	0	0	0	0	0	0	0	0	T_6	0	0	0
t_{14}	0	0	0	0	0	0	0	0	$1-T_{14}$	0	0	0	0	1	0	0
t_{15}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	T_{10}	0
t_{16}	0	0	0	0	$1-T_{16}$	0	0	0	0	0	0	0	0	0	0	1

$\text{Do}[\gamma = \gamma \text{ // } m_{1k\to 1},\ \{k, 2, 10\}];\ \gamma$

$\frac{T_1^2+T_{16}-T_1 T_{16}}{T_1^2}$	h_1	h_{11}	h_{12}	h_{13}	h_{14}	h_{15}	h_{16}
t_1	$\frac{T_{14}\left(-T_1+T_1^2+T_{16}\right)}{T_{12}\left(T_1^2+T_{16}-T_1 T_{16}\right)}$	$\frac{(-1+T_1)\left(1-T_1+T_1^2\right) T_{14} T_{16}}{T_1\left(T_1^2+T_{16}-T_1 T_{16}\right)}$	0	$-\frac{(-1+T_1)\left(1-T_1+T_1^2\right) T_{14}}{T_1^2+T_{16}-T_1 T_{16}}$	0	$1-T_1$	0
t_{11}	0	$\frac{1}{T_1}$	0	0	0	0	0
t_{12}	$\frac{-1+T_{12}}{T_{12}}$	0	1	0	0	0	0
t_{13}	0	0	0	T_1	0	0	0
t_{14}	$-\frac{(-1+T_{14})\left(-T_1+T_1^2+T_{16}\right)}{T_{12}\left(T_1^2+T_{16}-T_1 T_{16}\right)}$	$-\frac{(-1+T_1)\left(1-T_1+T_1^2\right)(-1+T_{14}) T_{16}}{T_1\left(T_1^2+T_{16}-T_1 T_{16}\right)}$	0	$\frac{(-1+T_1)\left(1-T_1+T_1^2\right)(-1+T_{14})}{T_1^2+T_{16}-T_1 T_{16}}$	1	0	0
t_{15}	0	0	0	0	0	T_1	0
t_{16}	$-\frac{T_1(-1+T_{16})}{T_{12}\left(T_1^2+T_{16}-T_1 T_{16}\right)}$	$-\frac{(-1+T_1) T_1 (-1+T_{16})}{T_1^2+T_{16}-T_1 T_{16}}$	0	$\frac{(-1+T_1)^2(-1+T_{16})}{T_1^2+T_{16}-T_1 T_{16}}$	0	0	1

8_17

$\gamma = Rm_{12,1}\, Rm_{27}\, Rm_{83}\, Rm_{4,11}\, Rp_{16,5}\, Rp_{6,13}\, Rp_{14,9}\, Rp_{10,15};$
$\text{Do}[\gamma = \gamma \text{ // } m_{1k\to 1},\ \{k, 2, 16\}];\ \gamma$

8_17

$-\frac{1-4 T_1+8 T_1^2-11 T_1^3+8 T_1^4-4 T_1^5+T_1^6}{T_1^3}$	h_1
t_1	1

$\gamma = Rm_{16}\, Rp_{74}\, Rp_{52}\, Rm_{8,11}\, Rp_{12,9}\, Rp_{10,3}$

1	h_1	h_2	h_3	h_4	h_5	h_6	h_7	h_8	h_9	h_{10}	h_{11}	h_{12}
t_1	1	0	0	0	0	$\frac{-1+T_1}{T_1}$	0	0	0	0	0	0
t_2	0	T_5	0	0	0	0	0	0	0	0	0	0
t_3	0	0	T_{10}	0	0	0	0	0	0	0	0	0
t_4	0	0	0	T_7	0	0	0	0	0	0	0	0
t_5	0	$1-T_5$	0	0	1	0	0	0	0	0	0	0
t_6	0	0	0	0	0	$\frac{1}{T_1}$	0	0	0	0	0	0
t_7	0	0	0	$1-T_7$	0	0	1	0	0	0	0	0
t_8	0	0	0	0	0	0	0	1	0	0	$\frac{-1+T_8}{T_8}$	0
t_9	0	0	0	0	0	0	0	0	T_{12}	0	0	0
t_{10}	0	0	$1-T_{10}$	0	0	0	0	0	0	1	0	0
t_{11}	0	0	0	0	0	0	0	0	0	0	$\frac{1}{T_8}$	0
t_{12}	0	0	0	0	0	0	0	0	$1-T_{12}$	0	0	1

$\gamma \text{ // } m_{14\to 1} \text{ // } m_{15\to 1} \text{ // } m_{16\to 1} \text{ // } m_{27\to 2} \text{ // } m_{28\to 2} \text{ // } m_{29\to 2} \text{ // } m_{2,10\to 2} \text{ // } m_{2,11\to 2} \text{ // } m_{3,12\to 3}$

$\frac{(-T_1-T_2+T_1 T_2)(-T_2-T_3+T_2 T_3)}{T_1 T_2}$	h_1	h_2	h_3
t_1	$-\frac{T_2}{-T_1-T_2+T_1 T_2}$	$\frac{-1+T_1}{-T_1-T_2+T_1 T_2}$	0
t_2	$-\frac{T_1(-1+T_2) T_3}{(-T_1-T_2+T_1 T_2)(-T_2-T_3+T_2 T_3)}$	$-\frac{(1-2 T_1-T_2+T_1 T_2) T_3}{(-T_1-T_2+T_1 T_2)(-T_2-T_3+T_2 T_3)}$	$\frac{-1+T_2}{-T_2-T_3+T_2 T_3}$
t_3	$\frac{T_1(-1+T_2) T_2(-1+T_3)}{(-T_1-T_2+T_1 T_2)(-T_2-T_3+T_2 T_3)}$	$\frac{T_2(1-2 T_1-T_2+T_1 T_2)(-1+T_3)}{(-T_1-T_2+T_1 T_2)(-T_2-T_3+T_2 T_3)}$	$\frac{1-2 T_2-T_3+T_2 T_3}{-T_2-T_3+T_2 T_3}$

$$\gamma = \mathrm{Rp}_{16}\,\mathrm{Rm}_{74}\,\mathrm{Rm}_{52}\,\mathrm{Rp}_{8,11}\,\mathrm{Rm}_{12,9}\,\mathrm{Rm}_{10,3};$$

$$\gamma \,//\, m_{14\to 1} \,//\, m_{15\to 1} \,//\, m_{16\to 1} \,//\, m_{27\to 2} \,//\, m_{28\to 2} \,//\, m_{29\to 2} \,//\, m_{2,10\to 2} \,//\, m_{2,11\to 2} \,//\, m_{3,12\to 3}$$

$$\begin{pmatrix} \frac{(-1+T_1+T_2)\,(-1+T_2+T_3)}{T_2\,T_3} & h_1 & h_2 & h_3 \\ t_1 & \frac{T_1}{-1+T_1+T_2} & \frac{(-1+T_1)\,T_2}{-1+T_1+T_2} & 0 \\ t_2 & \frac{(-1+T_2)\,T_2}{(-1+T_1+T_2)\,(-1+T_2+T_3)} & -\frac{T_2\,(1-T_1-2\,T_2+T_1\,T_2)}{(-1+T_1+T_2)\,(-1+T_2+T_3)} & \frac{(-1+T_2)\,T_3}{-1+T_2+T_3} \\ t_3 & \frac{(-1+T_2)\,(-1+T_3)}{(-1+T_1+T_2)\,(-1+T_2+T_3)} & -\frac{(1-T_1-2\,T_2+T_1\,T_2)\,(-1+T_3)}{(-1+T_1+T_2)\,(-1+T_2+T_3)} & -\frac{1-T_2-2\,T_3+T_2\,T_3}{-1+T_2+T_3} \end{pmatrix}$$

$$\gamma = \mathrm{Rm}_{16}\,\mathrm{Rp}_{74}\,\mathrm{Rp}_{52}\,\mathrm{Rm}_{8,11}\,\mathrm{Rp}_{12,9}\,\mathrm{Rp}_{10,3};$$

$$\gamma \,//\, m_{41\to 1} \,//\, m_{51\to 1} \,//\, m_{61\to 1} \,//\, m_{72\to 2} \,//\, m_{82\to 2} \,//\, m_{92\to 2} \,//\, m_{10,2\to 2} \,//\, m_{11,2\to 2} \,//\, m_{12,3\to 3}$$

$$\begin{pmatrix} (-T_1-T_2+T_1\,T_2)\,(-T_2-T_3+T_2\,T_3) & h_1 & h_2 & h_3 \\ t_1 & \frac{1-T_1-2\,T_2+T_1\,T_2}{-T_1-T_2+T_1\,T_2} & \frac{(-1+T_1)\,T_2\,(1-T_2-2\,T_3+T_2\,T_3)}{(-T_1-T_2+T_1\,T_2)\,(-T_2-T_3+T_2\,T_3)} & \frac{(-1+T_1)\,(-1+T_2)\,T_2\,T_3}{(-T_1-T_2+T_1\,T_2)\,(-T_2-T_3+T_2\,T_3)} \\ t_2 & \frac{-1+T_2}{-T_1-T_2+T_1\,T_2} & -\frac{T_1\,(1-T_2-2\,T_3+T_2\,T_3)}{(-T_1-T_2+T_1\,T_2)\,(-T_2-T_3+T_2\,T_3)} & -\frac{T_1\,(-1+T_2)\,T_3}{(-T_1-T_2+T_1\,T_2)\,(-T_2-T_3+T_2\,T_3)} \\ t_3 & 0 & \frac{-1+T_3}{-T_2-T_3+T_2\,T_3} & -\frac{T_2}{-T_2-T_3+T_2\,T_3} \end{pmatrix}$$

$$\gamma = \mathrm{Rp}_{16}\,\mathrm{Rm}_{74}\,\mathrm{Rm}_{52}\,\mathrm{Rp}_{8,11}\,\mathrm{Rm}_{12,9}\,\mathrm{Rm}_{10,3};$$

$$\gamma 1 = \gamma \,//\, m_{41\to 1} \,//\, m_{51\to 1} \,//\, m_{61\to 1} \,//\, m_{72\to 2} \,//\, m_{82\to 2} \,//\, m_{92\to 2} \,//\, m_{10,2\to 2} \,//\, m_{11,2\to 2} \,//\, m_{12,3\to 3}$$

$$\begin{pmatrix} \frac{(-1+T_1+T_2)\,(-1+T_2+T_3)}{T_1\,T_2^2\,T_3} & h_1 & h_2 & h_3 \\ t_1 & -\frac{1-2\,T_1-T_2+T_1\,T_2}{-1+T_1+T_2} & -\frac{(-1+T_1)\,(1-2\,T_2-T_3+T_2\,T_3)}{(-1+T_1+T_2)\,(-1+T_2+T_3)} & \frac{(-1+T_1)\,(-1+T_2)}{(-1+T_1+T_2)\,(-1+T_2+T_3)} \\ t_2 & \frac{T_1\,(-1+T_2)}{-1+T_1+T_2} & -\frac{T_2\,(1-2\,T_2-T_3+T_2\,T_3)}{(-1+T_1+T_2)\,(-1+T_2+T_3)} & \frac{(-1+T_2)\,T_2}{(-1+T_1+T_2)\,(-1+T_2+T_3)} \\ t_3 & 0 & \frac{T_2\,(-1+T_3)}{-1+T_2+T_3} & \frac{T_3}{-1+T_2+T_3} \end{pmatrix}$$

$$\{\gamma 1 \,//\, m_{23\to 2},\ \gamma 1 \,//\, m_{32\to 2}\}$$

$$\left\{\begin{pmatrix} \frac{1-T_1-2\,T_2+2\,T_1\,T_2+T_2^2}{T_1\,T_2^3} & h_1 & h_2 \\ t_1 & -\frac{-1+T_1+2\,T_2-3\,T_1\,T_2-T_2^2+T_1\,T_2^2}{1-T_1-2\,T_2+2\,T_1\,T_2+T_2^2} & -\frac{(-1+T_1)\,\left(1-3\,T_2+T_2^2\right)}{1-T_1-2\,T_2+2\,T_1\,T_2+T_2^2} \\ t_2 & \frac{T_1\,(-1+T_2)\,T_2}{1-T_1-2\,T_2+2\,T_1\,T_2+T_2^2} & \frac{T_2\,(1-T_1+T_1\,T_2)}{1-T_1-2\,T_2+2\,T_1\,T_2+T_2^2} \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} -\frac{(-1+T_1+T_2)\,\left(1-3\,T_2+T_2^2\right)}{T_1\,T_2^3} & h_1 & h_2 \\ t_1 & -\frac{1-2\,T_1-T_2+T_1\,T_2}{-1+T_1+T_2} & \frac{-1+T_1}{-1+T_1+T_2} \\ t_2 & \frac{T_1\,(-1+T_2)}{-1+T_1+T_2} & \frac{T_2}{-1+T_1+T_2} \end{pmatrix}\right\}$$

$$\begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \Xi \end{pmatrix} = \begin{pmatrix} -\frac{1-2\,T_1-T_2+T_1\,T_2}{-1+T_1+T_2} & -\frac{(-1+T_1)\,(1-2\,T_2-T_3+T_2\,T_3)}{(-1+T_1+T_2)\,(-1+T_2+T_3)} & \frac{(-1+T_1)\,(-1+T_2)}{(-1+T_1+T_2)\,(-1+T_2+T_3)} \\ \frac{T_1\,(-1+T_2)}{-1+T_1+T_2} & -\frac{T_2\,(1-2\,T_2-T_3+T_2\,T_3)}{(-1+T_1+T_2)\,(-1+T_2+T_3)} & \frac{(-1+T_2)\,T_2}{(-1+T_1+T_2)\,(-1+T_2+T_3)} \\ 0 & \frac{T_2\,(-1+T_3)}{-1+T_2+T_3} & \frac{T_3}{-1+T_2+T_3} \end{pmatrix};$$

$$\omega = \frac{(-1+T_1+T_2)\,(-1+T_2+T_3)}{T_1\,T_2^2\,T_3};$$

```
-β (ϵ + θ) + γ (ϵ + θ) + ϵ ϕ - θ ψ /. T3 → T2 // Simplify
```

$$\frac{(1+T_1(-2+T_2))(-1+T_2)}{(-1+T_1+T_2)(-1+2T_2)}$$

$$\begin{pmatrix} -\frac{1-2T_1-T_2+T_1T_2}{-1+T_1+T_2} & -\frac{(-1+T_1)(1-2T_2-T_3+T_2T_3)}{(-1+T_1+T_2)(-1+T_2+T_3)} & \frac{(-1+T_1)(-1+T_2)}{(-1+T_1+T_2)(-1+T_2+T_3)} \\ \frac{T_1(-1+T_2)}{-1+T_1+T_2} & -\frac{T_2(1-2T_2-T_3+T_2T_3)}{(-1+T_1+T_2)(-1+T_2+T_3)} & \frac{(-1+T_2)T_2}{(-1+T_1+T_2)(-1+T_2+T_3)} \\ 0 & \frac{T_2(-1+T_3)}{-1+T_2+T_3} & \frac{T_3}{-1+T_2+T_3} \end{pmatrix}.\{1,1,1\} \text{ // FullSimplify}$$

$$\left\{\frac{3-2T_3+T_2(-5+T_2+2T_3)-T_1(4-3T_3+T_2(-6+T_2+2T_3))}{(-1+T_1+T_2)(-1+T_2+T_3)}, \frac{T_1(-1+T_2)(-1+T_2+T_3)+T_2(-2-T_2(-3+T_3)+T_3)}{(-1+T_1+T_2)(-1+T_2+T_3)}, \frac{T_2(-1+T_3)+T_3}{-1+T_2+T_3}\right\}$$

$$\{1,1,1\}.\begin{pmatrix} -\frac{1-2T_1-T_2+T_1T_2}{-1+T_1+T_2} & -\frac{(-1+T_1)(1-2T_2-T_3+T_2T_3)}{(-1+T_1+T_2)(-1+T_2+T_3)} & \frac{(-1+T_1)(-1+T_2)}{(-1+T_1+T_2)(-1+T_2+T_3)} \\ \frac{T_1(-1+T_2)}{-1+T_1+T_2} & -\frac{T_2(1-2T_2-T_3+T_2T_3)}{(-1+T_1+T_2)(-1+T_2+T_3)} & \frac{(-1+T_2)T_2}{(-1+T_1+T_2)(-1+T_2+T_3)} \\ 0 & \frac{T_2(-1+T_3)}{-1+T_2+T_3} & \frac{T_3}{-1+T_2+T_3} \end{pmatrix} \text{ // FullSimplify}$$

$$\{1,1,1\}$$

$$\text{Clear}[\alpha,\beta,\gamma,\delta,\theta,\epsilon,\phi,\psi,\Xi,\mu,\omega];\ \Gamma\left[\omega,\{t_a,t_b,t_S\}.\begin{pmatrix}\alpha&\beta&\theta\\\gamma&\delta&\epsilon\\\phi&\psi&\Xi\end{pmatrix}.\{h_a,h_b,h_S\}\right]$$

$$\begin{pmatrix}\omega & h_a & h_b & h_S\\ t_a & \alpha & \beta & \theta\\ t_b & \gamma & \delta & \epsilon\\ t_S & \phi & \psi & \Xi\end{pmatrix}$$

$$\Gamma\left[\omega,\{t_a,t_b,t_S\}.\begin{pmatrix}\alpha&\beta&\theta\\\gamma&\delta&\epsilon\\\phi&\psi&\Xi\end{pmatrix}.\{h_a,h_b,h_S\}\right] \text{ // } m_{ab\to c}$$

$$\begin{pmatrix}-(-1+\beta)\,\omega & h_c & h_S\\ t_c & \frac{-\gamma+\beta\gamma-\alpha\delta}{-1+\beta} & \frac{-\epsilon+\beta\epsilon-\delta\theta}{-1+\beta}\\ t_S & \frac{-\phi+\beta\phi-\alpha\psi}{-1+\beta} & \frac{-\Xi+\beta\Xi-\theta\psi}{-1+\beta}\end{pmatrix}$$

$$\Gamma\left[\omega,\{t_a,t_b,t_S\}.\begin{pmatrix}\alpha&\beta&\theta\\\gamma&\delta&\epsilon\\\phi&\psi&\Xi\end{pmatrix}.\{h_a,h_b,h_S\}\right] \text{ // } m_{ba\to c}$$

$$\begin{pmatrix}-(-1+\gamma)\,\omega & h_c & h_S\\ t_c & \frac{-\beta+\beta\gamma-\alpha\delta}{-1+\gamma} & \frac{-\alpha\epsilon-\theta+\gamma\theta}{-1+\gamma}\\ t_S & \frac{-\delta\phi-\psi+\gamma\psi}{-1+\gamma} & \frac{-\Xi+\gamma\Xi-\epsilon\phi}{-1+\gamma}\end{pmatrix}$$

```
(1 - γ) ( (-β+β γ-α δ)/(-1+γ)   (-α ϵ-θ+γ θ)/(-1+γ)
          (-δ ϕ-ψ+γ ψ)/(-1+γ)   (-Ξ+γ Ξ-ϵ ϕ)/(-1+γ) ) // Simplify // MatrixForm
```

$$\begin{pmatrix} \beta - \beta\,\gamma + \alpha\,\delta & \alpha \in + \Theta - \gamma\,\Theta \\ \delta\,\phi + \psi - \gamma\,\psi & \Xi - \gamma\,\Xi + \in \phi \end{pmatrix}$$

```
FullSimplify[(Ξ + ϵ ϕ / (1 - γ) - 1) (1 - γ) - (Ξ + θ ψ / (1 - β) - 1) (1 - β) /. {Ξ → 1 - ϵ - θ}]
```

$$-\beta\,(\in + \Theta) + \gamma\,(\in + \Theta) + \in \phi - \Theta\,\psi$$

```
FullSimplify[(Ξ + ϵ ϕ / (1 - γ) - 1) - (Ξ + θ ψ / (1 - β) - 1) /. {Ξ → 1 - ϵ - θ}]
```

$$\frac{\in \phi}{1-\gamma} + \frac{\Theta\,\psi}{-1+\beta}$$

```
ΓZ[L_] := Module[{s, Z, c, k},
   s = Skeleton[L];
   Z =
    Times @@ PD[L] /. X[i_, j_, k_, l_] :→ If[PositiveQ[X[i, j, k, l]], Rp_{l i}, Rm_{j i}];
   Do[Z = Z // m_{s⟦c,1⟧, s⟦c,k⟧ → s⟦c,1⟧}, {c, Length[s]}, {k, 2, Length[s⟦c⟧]}];
   Z];
ΓA[K_] := ΓZ[K]⟦1⟧ /. T_ :→ T;
ΓMVA[L_Link] := Module[{Hs, ω, μ, A},
  {ω, μ} = List @@ ΓZ[L];
  Hs = Rest[h_# & /@ (First /@ Skeleton[L])];
  A = Outer[Coefficient[μ, #1 * #2] &, Hs, Hs /. h_a_ :→ t_a];
  Factor[ ω Det[A - IdentityMatrix@Length@Hs] / (1 - T_{Skeleton[L]⟦1,1⟧}) ]
 ]
```

```
ΓZ[Link["L6a4"]]
```

KnotTheory::loading : Loading precomputed data in PD4Links`.

$$\left(\begin{array}{cc} \frac{(1-T_1-T_5+T_1 T_5-T_9+T_1 T_9+T_5 T_9)\,(T_1+T_5-T_1 T_5+T_9-T_1 T_9-T_5 T_9+T_1 T_5 T_9)}{T_1 T_5 T_9} & h_1 \\ t_1 & \frac{T_9\,(1-2 T_1+T_1^2-T_5+3 T_1 T_5-T_1^2 T_5-T_9+2 T_1 T_9-T_1^2 T_9+T_5 T_9-2 T_1 T_5 T_9+\cdots}{(1-T_1-T_5+T_1 T_5-T_9+T_1 T_9+T_5 T_9)\,(T_1+T_5-T_1 T_5+T_9-T_1 T_9-T_5 T_9+T_1\cdots} \\ t_5 & \frac{T_1\,(-1+T_5)\,(1-T_1+T_1 T_5)\,(-1+T_9)}{(1-T_1-T_5+T_1 T_5-T_9+T_1 T_9+T_5 T_9)\,(T_1+T_5-T_1 T_5+T_9-T_1 T_9-T_5 T_9+T_1\cdots} \\ t_9 & \frac{(-1+T_5)\,T_5\,(-1+T_9)\,(1-2 T_1-T_9+T_1 T_9)}{(1-T_1-T_5+T_1 T_5-T_9+T_1 T_9+T_5 T_9)\,(T_1+T_5-T_1 T_5+T_9-T_1 T_9-T_5 T_9+T_1\cdots} \end{array}\right.$$

```
ΓA[Knot[8, 17]]
```

KnotTheory::loading : Loading precomputed data in PD4Knots`.

$$-\frac{1 - 4\,T + 8\,T^2 - 11\,T^3 + 8\,T^4 - 4\,T^5 + T^6}{T^2}$$

```
ΓMVA[Link["L6a4"]]
```

$$-\frac{(-1+T_1)\,(-1+T_5)\,(-1+T_9)}{T_1\,T_5}$$

```
Factor[Alexander[#][T] / ΓA[#]] & /@ AllKnots[{3, 9}]
```

$$\left\{T, \frac{1}{T}, T^2, T^2, 1, 1, 1, T^3, T^3, \frac{1}{T^4}, \frac{1}{T^4}, T^3, T, \frac{1}{T^2}, T, T, \frac{1}{T}, \frac{1}{T}, \frac{1}{T^3}, T, \frac{1}{T}, \frac{1}{T}, \frac{1}{T}, \frac{1}{T}, T, \frac{1}{T}, \frac{1}{T}, T, T^3, T, \frac{1}{T}, 1, \frac{1}{T^4}, 1, T, T^4, T^4, \frac{1}{T^5}, T^4, \frac{1}{T^5}, T^4, T^4, 1, T^4, \frac{1}{T^5}, \frac{1}{T^3}, T^2, \frac{1}{T^5}, \frac{1}{T^3}, \frac{1}{T^3}, \frac{1}{T^5}, \frac{1}{T}, T^4, \frac{1}{T}, T^2, \frac{1}{T^3}, \frac{1}{T^2}, T^4, 1, T^2, \frac{1}{T^3}, T, T, \frac{1}{T}, 1, T, \frac{1}{T^3}, 1, 1, T^4, \frac{1}{T^3}, \frac{1}{T}, T^4, \frac{1}{T^4}, 1, 1, \frac{1}{T^2}, \frac{1}{T^3}, 1, T^2, 1, \frac{1}{T^2}, \frac{1}{T^4}, \frac{1}{T^6}\right\}$$

```
Factor[1/ΓMVA[#] (MultivariableAlexander[#][T] /. T[i_] :> T_Skeleton[#][[i,1]])] & /@
 AllLinks[{2, 8}]
```

KnotTheory::loading : Loading precomputed data in MultivariableAlexander4Links`.

$$\left\{-T_1^2 T_3, -T_1^{3/2} T_5^{3/2}, -\sqrt{T_1}\, T_5^{3/2}, -T_1^{3/2} \sqrt{T_5}, -T_1^2 T_7^2, -T_1^2 T_7^2, -\frac{\sqrt{T_1}\sqrt{T_5}}{\sqrt{T_9}}, -T_1^{3/2} T_5^{3/2} T_9^{3/2}, -\frac{\sqrt{T_1}\sqrt{T_5}}{T_9^{3/2}}, -\sqrt{T_1}\sqrt{T_5}, -T_1^{3/2} T_5^{7/2}, -\frac{\sqrt{T_1}}{T_5^{3/2}}, -\frac{\sqrt{T_1}}{T_5^{3/2}}, -T_1 T_7^2, -\frac{1}{T_7}, -\frac{T_1^{3/2}\sqrt{T_5}}{\sqrt{T_9}}, -T_1^{3/2} T_5^{7/2}, -\sqrt{T_1}\, T_5^{5/2}, -\sqrt{T_1}\, T_5^{3/2}, -\frac{\sqrt{T_1}}{\sqrt{T_5}}, -T_1^{3/2} T_5^{3/2}, -\sqrt{T_1}\, T_5^{3/2}, -\frac{T_1^{3/2}}{\sqrt{T_5}}, -\frac{T_1^{3/2}}{\sqrt{T_5}}, -T_1^{3/2} T_5^{7/2}, -\frac{T_1}{T_7}, -T_1 T_7, -T_1^2 T_7^3, -T_1^2 T_7^3, -T_1^{5/2} T_9^{5/2}, -T_1^{5/2} T_9^{5/2}, -T_1^{5/2} T_9^{5/2}, -T_1^{3/2} T_5^{3/2} \sqrt{T_9}, -\sqrt{T_1}, -T_1^{3/2} T_5^2 T_{11}^2, -\frac{\sqrt{T_1}}{T_{11}^2}, -\frac{\sqrt{T_1}\, T_5}{T_{11}}, -\frac{T_1^{3/2}\sqrt{T_5}}{T_{13}^{3/2}}, -T_1^{3/2} T_5^{3/2} T_9^{3/2} T_{13}^{3/2}, -T_1^{3/2} T_5^{3/2}, -\sqrt{T_1}\sqrt{T_5}, -T_1^{3/2} T_5^2 T_{11}^2, -\sqrt{T_1}\, T_5^2 T_{11}, -\sqrt{T_1}\, T_5^{3/2} T_{11}^{3/2}, -T_1^{3/2} T_5^{5/2} \sqrt{T_{13}}, -\frac{\sqrt{T_1}\sqrt{T_5}}{\sqrt{T_9}\sqrt{T_{13}}}, -\sqrt{T_1}\sqrt{T_5}\sqrt{T_9}\sqrt{T_{13}}\right\}$$