Dror Bar-Natan: Talks: Bannoye-1407:

The Kashiwara-Vergne Problem and Topology

ω :=http://www.math.toronto.edu/~drorbn/Talks/Bannoye-1407
Video, handout, links at ω/

Abstract. I will describe a general machine, a close cousin of Taylor's theorem, whose inputs are topics in topology and whose outputs are problems in algebra. There are many inputs the machine can take, and many outputs it produces, but I will concentrate on just one input/output pair. When fed with a certain class of knotted 2-dimensional objects in 4-dimensional space, it outputs the Kashiwara-Vergne Problem (1978 ω/KV, solved Alekseev-Meinrenken 2006 ω/AM, elucidated Alekseev-Torossian 2008-2012 ω/AT), a problem about convolutions on Lie groups and Lie algebras.

The Kashiwara-Vergne Conjecture. There exist two series F and G in the completed free Lie algebra FL in generators x and y so that

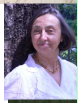
$$x+y-\log e^y e^x = (1-e^{-\operatorname{ad} x})F + (e^{\operatorname{ad} y}-1)G \text{ in } FL$$

and so that with $z = \log e^x e^y$,

$$\operatorname{tr}(\operatorname{ad} x)\partial_x F + \operatorname{tr}(\operatorname{ad} y)\partial_y G \quad \text{in cyclic words}$$
$$= \frac{1}{2}\operatorname{tr}\left(\frac{\operatorname{ad} x}{e^{\operatorname{ad} x}-1} + \frac{\operatorname{ad} y}{e^{\operatorname{ad} y}-1} - \frac{\operatorname{ad} z}{e^{\operatorname{ad} z}-1} - 1\right)$$

Implies the loosely-stated convolutions statement: Convolutions of invariant functions on a Lie group agree with convolutions of invariant functions on its Lie algebra.


Kashiwara


Vergne


Alekseev


Meinrenken


Torossian

The Machine. Let G be a group, $\mathcal{K} = \mathbb{Q}G = \{\sum a_i g_i \colon a_i \in \mathbb{Q},\, g_i \in G\}$ its group-ring, $\mathcal{I} = \{\sum a_i g_i \colon \sum a_i = 0\} \subset \mathcal{K}$ its augmentation ideal. Let

$$\mathcal{A} = \operatorname{gr}\mathcal{K} := \widehat{\bigoplus_{m\geq 0}} \mathcal{I}^m/\mathcal{I}^{m+1}.$$

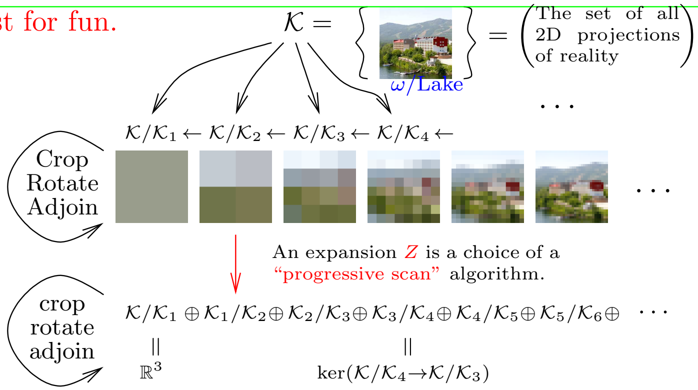
P.S. $(\mathcal{K}/\mathcal{I}^{m+1})^*$ is Vassiliev / finite-type / polynomial invariants.

Note that $\mathcal{A}$ inherits a product from G.

Definition. A linear $Z\colon \mathcal{K} \to \mathcal{A}$ is an "expansion" if for any $\gamma \in \mathcal{I}^m$, $Z(\gamma) = (0, \dots, 0, \gamma/\mathcal{I}^{m+1}, *, \dots)$, and a "homomorphic expansion" if in addition it preserves the product.

Example. Let $\mathcal{K} = C^\infty(\mathbb{R}^n)$ and $\mathcal{I} = \{f \colon f(0) = 0\}$. Then $\mathcal{I}^m = \{f \colon f \text{ vanishes like } |x|^m\}$ so $\mathcal{I}^m/\mathcal{I}^{m+1}$ degree m homogeneous polynomials and $\mathcal{A} = \{\text{power series}\}$. The Taylor series is a homomorphic expansion!

Just for fun.


$\mathcal{K} = \{\ \}$ = (The set of all 2D projections of reality)

ω/Lake

$\mathcal{K}/\mathcal{K}_1 \leftarrow \mathcal{K}/\mathcal{K}_2 \leftarrow \mathcal{K}/\mathcal{K}_3 \leftarrow \mathcal{K}/\mathcal{K}_4 \leftarrow$ …

Crop Rotate Adjoin

An expansion Z is a choice of a "progressive scan" algorithm.

$\mathcal{K}/\mathcal{K}_1 \oplus \mathcal{K}_1/\mathcal{K}_2 \oplus \mathcal{K}_2/\mathcal{K}_3 \oplus \mathcal{K}_3/\mathcal{K}_4 \oplus \mathcal{K}_4/\mathcal{K}_5 \oplus \mathcal{K}_5/\mathcal{K}_6 \oplus \cdots$

$\mathcal{K}/\mathcal{K}_1 = \mathbb{R}^3$, $\mathcal{K}_3/\mathcal{K}_4 = \ker(\mathcal{K}/\mathcal{K}_4 \to \mathcal{K}/\mathcal{K}_3)$

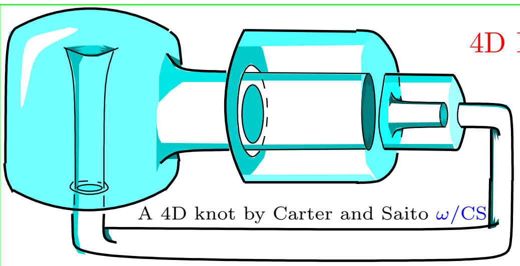
crop rotate adjoin

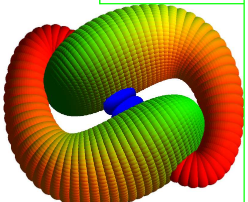
In the finitely presented case, finding Z amounts to solving a system of equations in a graded space.

Theorem (with Zsuzsanna Dancso, ω/WKO). There is a bijection between the set of homomorphic expansions for $w\mathcal{K}$ and the set of solutions of the Kashiwara-Vergne problem. This is the tip of a major iceberg.

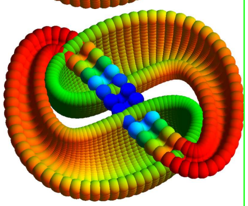

Dancso, ω/ZD

4D Knots.


A 4D knot by Carter and Saito ω/CS

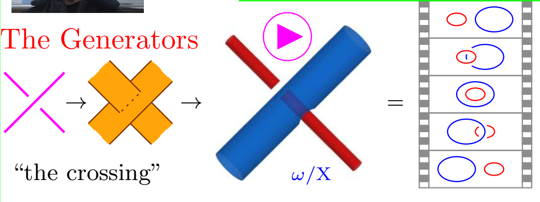



Dalvit ω/Dal

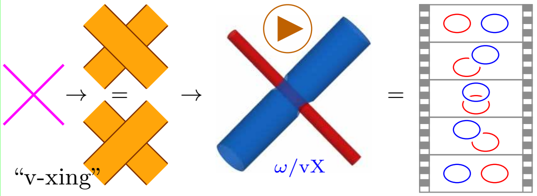



Satoh

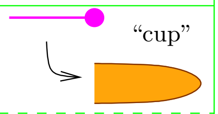
The Generators

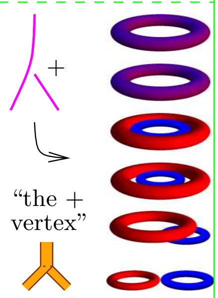

"the crossing"

ω/X

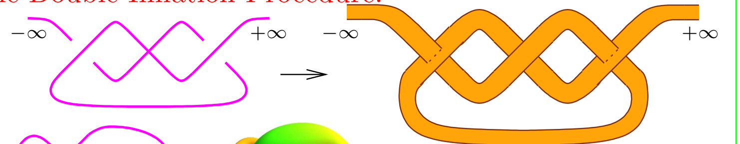

"v-xing"

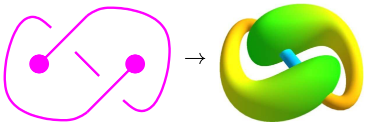
ω/vX

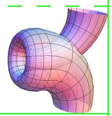

"cup"


"the + vertex"

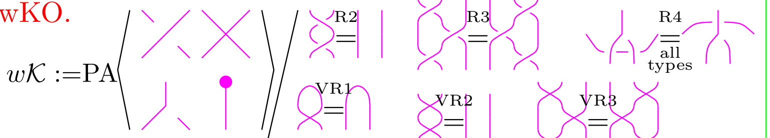
The Double Inflation Procedure.



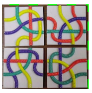


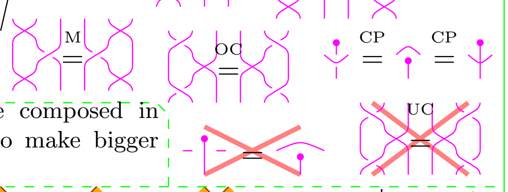

Riddle. What band, inflated, gives the "Wen"?

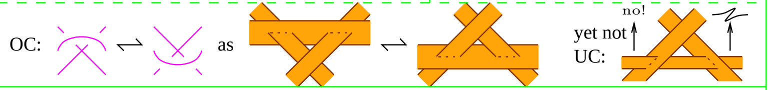
wKO.

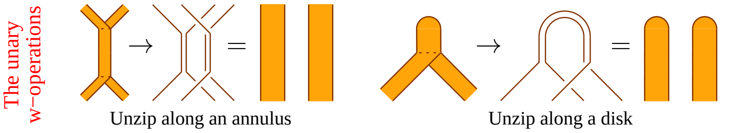

$w\mathcal{K} :=$PA$\langle\ \rangle$

"Planar Algebra": The objects are "tiles" that can be composed in arbitrary planar ways to make bigger tiles.

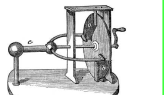


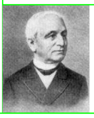






The Machine generalizes to arbitrary algebraic structures!


ω/mac



"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

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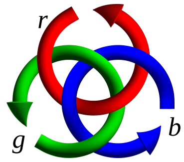
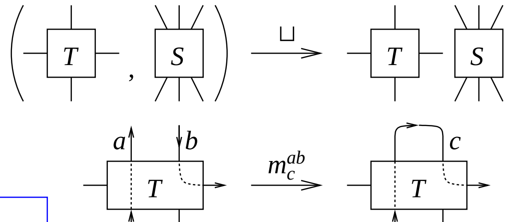
Some very good formulas for the Alexander polynomial, 1

Abstract. I will describe some very good formulas for a *(matrix plus scalar)*-valued extension of the Alexander polynomnial to tangles, then say that everything extends to virtual tangles, then roughly to simply knotted balloons and hoops in 4D, then the target space extends to *(free Lie algebras plus cyclic words)*, and the result is a universal finite type of the knotted objects in its domain. Taking a cue from the BF topological quantum field theory, everything should extend (with some modifications) to arbitrary codimension-2 knots in arbitrary dimension and in particular, to arbitrary 2-knots in 4D. But what is really going on is still a mystery.

Tangles.





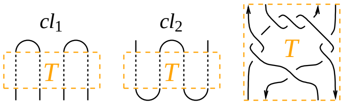
(meta-associativity: $m_a^{ab}/\!/m_a^{ac}=m_b^{bc}/\!/m_a^{ab}$)

Why Tangles?

- Finitely presented.
- Divide and conquer proofs and computations.
- "Algebraic Knot Theory": If K is ribbon,

$$Z(K)\in\{cl_2(Z)\colon cl_1(Z)=1\}.$$

(Genus and crossing number are also definable properties).



trivial ribbon example

Theorem 1. ∃! an invariant γ: {pure framed S-component tangles} → $R\times M_{S\times S}(R)$, where $R=R_S=\mathbb{Z}((T_a)_{a\in S})$ is the ring of rational functions in S variables, intertwining

$$\textbf{1.}\ \left(\begin{array}{c|c}\omega_1 & S_1\\\hline S_1 & A_1\end{array},\begin{array}{c|c}\omega_2 & S_2\\\hline S_2 & A_2\end{array}\right)\xrightarrow{\sqcup}\begin{array}{c|cc}\omega_1\omega_2 & S_1 & S_2\\\hline S_1 & A_1 & 0\\ S_2 & 0 & A_2\end{array},$$

$$\textbf{2.}\ \begin{array}{c|ccc}\omega & a & b & S\\\hline a & \alpha & \beta & \theta\\ b & \gamma & \delta & \epsilon\\ S & \phi & \psi & \Xi\end{array}\xrightarrow[\mu:=1-\beta]{m_c^{ab}}\left(\begin{array}{c|cc}\mu\omega & c & S\\\hline c & \gamma+\alpha\delta/\mu & \epsilon+\delta\theta/\mu\\ S & \phi+\alpha\psi/\mu & \Xi+\psi\theta/\mu\end{array}\right)_{T_a,T_b\to T_c},$$

and satisfying $\left(\uparrow_a;\ {}_a\nearrow\!\!\!\nwarrow_b,\ {}_b\nwarrow\!\!\!\nearrow_a\right)\xrightarrow{\gamma}\left(\begin{array}{c|c}1 & a\\\hline a & 1\end{array};\ \begin{array}{c|cc}1 & a & b\\\hline a & 1 & 1-T_a^{\pm1}\\ b & 0 & T_a^{\pm1}\end{array}\right).$

In Addition • The matrix part is just a stitching formula for Burau/Gassner [LD, KLW, CT].

- $L\mapsto\omega$ is Alexander, mod units.
- $L\mapsto(\omega,A)\mapsto\omega\det'(A-I)/(1-T')$ is the MVA, mod units.
- The "fastest" Alexander algorithm.
- There are also formulas for strand deletion, reversal, and doubling.
- Every step along the computation is the invariant of something.
- Extends to and more naturally defined on v/w-tangles.
- Fits in one column, including propaganda & implementation.

Implementation key idea:

$$(\omega,A=(\alpha_{ab}))\leftrightarrow$$
$$(\omega,\lambda=\textstyle\sum\alpha_{ab}t_ah_b)$$

ωεβ/Demo

```
ΓCollect[Γ[ω_, λ_]] := Γ[Factor[ω],
   Collect[λ, h_, Collect[#, t_, Factor] &]];
Format[Γ[ω_, λ_]] := Module[{S, M},
   S = Union@Cases[Γ[ω, λ], (h | t)a_ :→ a, ∞];
   M = Outer[Factor[∂h#1t#2 λ] &, S, S];
   M = Prepend[M, t# & /@ S] // Transpose;
   M = Prepend[M, Prepend[h# & /@ S, ω]];
   M // MatrixForm];
```

```
Γ /: Γ[ω1_, λ1_] Γ[ω2_, λ2_] := Γ[ω1 * ω2, λ1 + λ2];
ma_b_→c_[Γ[ω_, λ_]] := Module[{α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ, μ},
```

$$\begin{pmatrix}\alpha&\beta&\theta\\\gamma&\delta&\epsilon\\\phi&\psi&\Xi\end{pmatrix}=\begin{pmatrix}\partial_{t_a,h_a}\lambda&\partial_{t_a,h_b}\lambda&\partial_{t_a}\lambda\\\partial_{t_b,h_a}\lambda&\partial_{t_b,h_b}\lambda&\partial_{t_b}\lambda\\\partial_{h_a}\lambda&\partial_{h_b}\lambda&\lambda\end{pmatrix}\ /.\ (t\,|\,h)_{a|b}\to0;$$

$$\Gamma\Big[(\mu=1-\beta)\,\omega,\ \{t_c,1\}.\begin{pmatrix}\gamma+\alpha\delta/\mu&\epsilon+\delta\theta/\mu\\\phi+\alpha\psi/\mu&\Xi+\psi\theta/\mu\end{pmatrix}.\{h_c,1\}$$
$$\quad/.\ \{T_a\to T_c,\ T_b\to T_c\}\ /\!/\ \Gamma\mathrm{Collect}\Big];$$

$$\mathrm{Rp}_{a\_b\_}:=\Gamma\Big[1,\ \{t_a,t_b\}.\begin{pmatrix}1&1-T_a\\0&T_a\end{pmatrix}.\{h_a,h_b\}\Big];$$

$$\mathrm{Rm}_{a\_b\_}:=\mathrm{Rp}_{ab}\ /.\ T_a\to1/T_a;$$

Meta-Associativity

Runs.

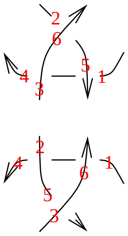
$$\gamma=\Gamma\left[\omega,\ \{t_1,t_2,t_3,t_s\}.\begin{pmatrix}\alpha_{11}&\alpha_{12}&\alpha_{13}&\theta_1\\\alpha_{21}&\alpha_{22}&\alpha_{23}&\theta_2\\\alpha_{31}&\alpha_{32}&\alpha_{33}&\theta_3\\\phi_1&\phi_2&\phi_3&\Xi\end{pmatrix}.\{h_1,h_2,h_3,h_s\}\right];$$

$$(\gamma\ /\!/\ m_{12\to1}\ /\!/\ m_{13\to1})==(\gamma\ /\!/\ m_{23\to2}\ /\!/\ m_{12\to1})$$

```
True
```

R3 … divide and conquer!

$$\{\mathrm{Rm}_{51}\,\mathrm{Rm}_{62}\,\mathrm{Rp}_{34}\ /\!/\ m_{14\to1}\ /\!/\ m_{25\to2}\ /\!/\ m_{36\to3},$$
$$\mathrm{Rp}_{61}\,\mathrm{Rm}_{24}\,\mathrm{Rm}_{35}\ /\!/\ m_{14\to1}\ /\!/\ m_{25\to2}\ /\!/\ m_{36\to3}\}$$



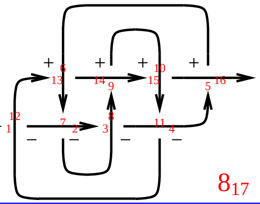
$$\left\{\begin{pmatrix}1&h_1&h_2&h_3\\t_1&\frac{T_3}{T_2}&0&0\\t_2&\frac{-1+T_2}{T_2}&\frac{1}{T_3}&0\\t_3&-\frac{-1+T_3}{T_2}&\frac{-1+T_3}{T_3}&1\end{pmatrix},\begin{pmatrix}1&h_1&h_2&h_3\\t_1&\frac{T_3}{T_2}&0&0\\t_2&\frac{-1+T_2}{T_2}&\frac{1}{T_3}&0\\t_3&-\frac{-1+T_3}{T_2}&\frac{-1+T_3}{T_3}&1\end{pmatrix}\right\}$$

$$\gamma=\mathrm{Rm}_{12,1}\,\mathrm{Rm}_{27}\,\mathrm{Rm}_{83}\,\mathrm{Rm}_{4,11}\,\mathrm{Rp}_{16,5}\,\mathrm{Rp}_{6,13}\,\mathrm{Rp}_{14,9}\,\mathrm{Rp}_{10,15};$$

$$\mathrm{Do}[\gamma=\gamma\ /\!/\ m_{1k\to1},\ \{k,2,16\}];$$

$$\gamma$$

$$\begin{pmatrix}-\frac{1-4T_1+8T_1^2-11T_1^3+8T_1^4-4T_1^5+T_1^6}{T_1^3}&h_1\\t_1&1\end{pmatrix}$$



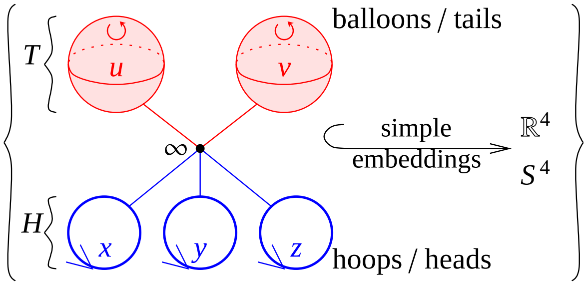
8_{17}

Weaknesses, • m_c^{ab} is non-linear.

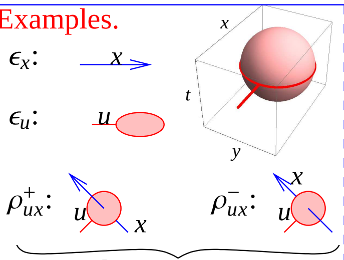
- The product ωA is always Laurent, but proving this takes induction with exponentially many conditions.

$\mathcal{K}^{bh}(H;T)$.

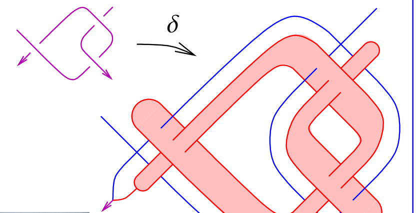
"Simply-knotted balloons and hoops"



Examples.



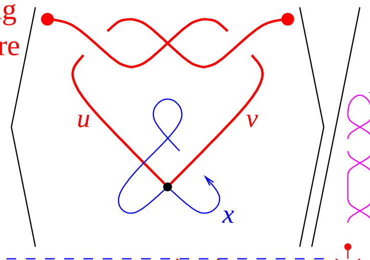
"the generators"

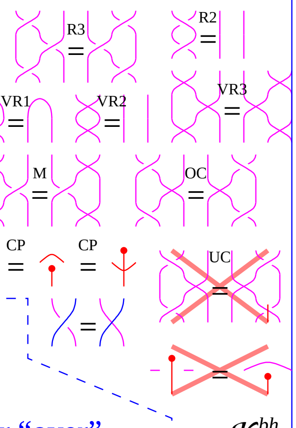


Shin Satoh

Disturbing Conjecture

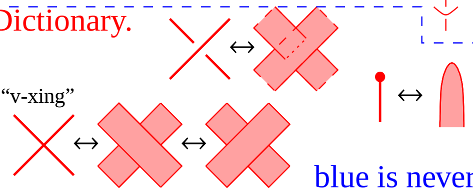
$$\mathcal{K}^{bh}\overset{?}{=}$$





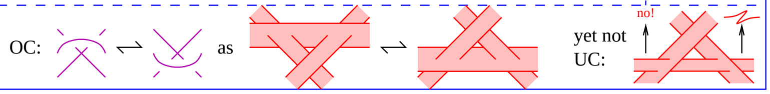
Dictionary.

"v-xing"



blue is never "over"

=: $w\mathcal{K}^{bh}$



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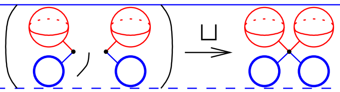
Some very good formulas for the Alexander polynomial, 2

Operations

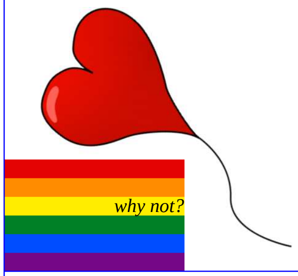
Punctures & Cuts

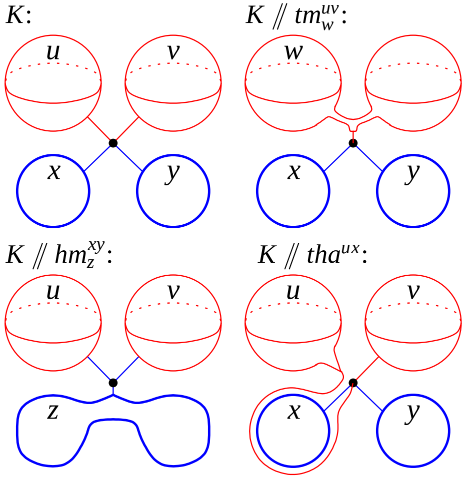
Connected Sums.



If X is a space, $\pi_1(X)$ is a group, $\pi_2(X)$ is an Abelian group, and π_1 acts on π_2.

Proposition. The generators generate.





Definition. l_{xu} is the linking number of hoop x with balloon u. For $x \in H$, $\sigma_x := \prod_{u\in T} T_u^{l_{xu}} \in R = R_T = \mathbb{Z}((T_a)_{a\in T})$, the ring of rational functions in T variables.

Theorem 2 [BNS]. ∃! an invariant $\beta\colon w\mathcal{K}^{bh}(H;T) \to R \times M_{T\times H}(R)$, intertwining

$$1.\ \left(\begin{array}{c|c}\omega_1 & H_1\\ \hline T_1 & A_1\end{array}, \begin{array}{c|c}\omega_2 & H_2\\ \hline T_2 & A_2\end{array}\right) \xrightarrow{\sqcup} \begin{array}{c|cc}\omega_1\omega_2 & H_1 & H_2\\ \hline T_1 & A_1 & 0\\ T_2 & 0 & A_2\end{array},$$

$$2.\ \begin{array}{c|c}\omega & H\\ \hline u & \alpha\\ v & \beta\\ T & \Xi\end{array} \xrightarrow{tm^{uv}_w} \left(\begin{array}{c|c}\omega & H\\ \hline w & \alpha+\beta\\ T & \Xi\end{array}\right)_{T_u,T_v\to T_w},$$

$$3.\ \begin{array}{c|ccc}\omega & x & y & H\\ \hline T & \alpha & \beta & \Xi\end{array} \xrightarrow{hm^{xy}_z} \begin{array}{c|cc}\omega & z & H\\ \hline T & \alpha+\sigma_x\beta & \Xi\end{array},$$

$$4.\ \begin{array}{c|cc}\omega & x & H\\ \hline u & \alpha & \theta\\ T & \phi & \Xi\end{array} \xrightarrow[\nu:=1+\alpha]{tha^{ux}} \begin{array}{c|cc}\nu\omega & x & H\\ \hline u & \sigma_x\alpha/\nu & \sigma_x\theta/\nu\\ T & \phi/\nu & \Xi-\phi\theta/\nu\end{array},$$

and satisfying $(\epsilon_x;\epsilon_u;\rho^{\pm}_{ux}) \xrightarrow{\beta} \left(\begin{array}{c|c}1 & x\\ \hline & \end{array}; \begin{array}{c|c}1 & \\ \hline u & \end{array}; \begin{array}{c|c}1 & x\\ \hline u & T_u^{\pm1}-1\end{array}\right)$.

Proposition. If T is a u-tangle and $\beta(\delta T) = (\omega, A)$, then $\gamma(T) = (\omega, \sigma - A)$, where $\sigma = \operatorname{diag}(\sigma_a)_{a\in S}$. Under this, $m^{ab}_c \leftrightarrow tha^{ab} /\!\!/ tm^{ab}_c /\!\!/ hm^{ab}_c$.

References.

[BN] D. Bar-Natan, *Balloons and Hoops and their Universal Finite Type Invariant, BF Theory, and an Ultimate Alexander Invariant*, ωεβ/KBH, arXiv:1308.1721.

[BND] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects I-II*, ωεβ/WKO1, ωεβ/WKO2, arXiv:1405.1956, arXiv:1405.1955.

[BNS] D. Bar-Natan and S. Selmani, *Meta-Monoids, Meta-Bicrossed Products, and the Alexander Polynomial*, J. of Knot Theory and its Ramifications **22-10** (2013), arXiv:1302.5689.

[CR] A. S. Cattaneo and C. A. Rossi, *Wilson Surfaces and Higher Dimensional Knot Invariants*, Commun. in Math. Phys. **256-3** (2005) 513–537, arXiv:math-ph/0210037.

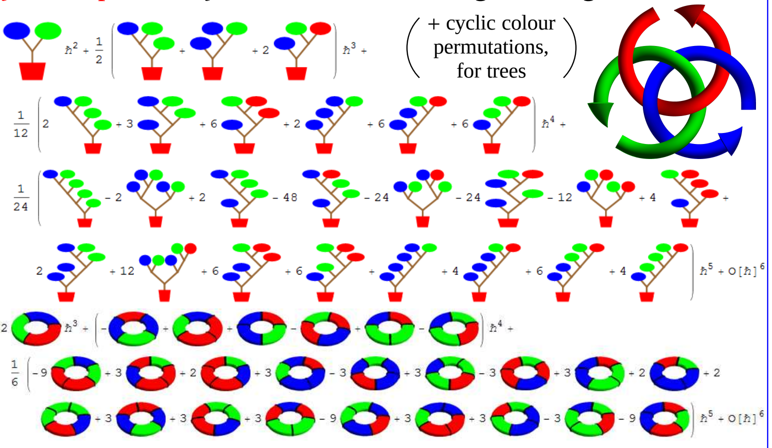
[CT] D. Cimasoni and V. Turaev, *A Lagrangian Representation of Tangles*, Topology **44** (2005) 747–767, arXiv:math.GT/0406269.

[KLW] P. Kirk, C. Livingston, and Z. Wang, *The Gassner Representation for String Links*, Comm. Cont. Math. **3** (2001) 87–136, arXiv:math/9806035.

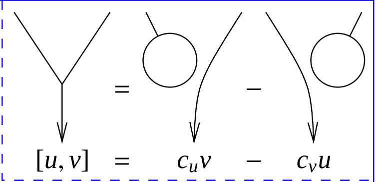
[LD] J. Y. Le Dimet, *Enlacements d'Intervalles et Représentation de Gassner*, Comment. Math. Helv. **67** (1992) 306–315.

Theorem 3 [BND, BN]. ∃! a homomorphic expansion, aka a homomorphic universal finite type invariant Z of w-knotted balloons and hoops. $\zeta := \log Z$ takes values in $FL(T)^H \times CW(T)$.

ζ is computable! ζ of the Borromean tangle, to degree 5:



Proposition [BN]. Modulo all relations that universally hold for the 2D non-Abelian Lie algebra and after some changes-of-variable, ζ reduces to β and the KBH operations on ζ reduce to the formulas in Theorem 2.



$$[u,v] = c_u v - c_v u$$

A Big Question. Does it all extend to arbitrary 2-knots (not necessarily "simple")? To arbitrary codimension-2 knots?

BF Following [CR]. $A \in \Omega^1(M=\mathbb{R}^4, \mathfrak{g})$, $B \in \Omega^2(M, \mathfrak{g}^*)$,

$$S(A,B) := \int_M \langle B, F_A\rangle.$$

Rossi

With $\kappa\colon (S=\mathbb{R}^2) \to M$, $\beta \in \Omega^0(S, \mathfrak{g})$, $\alpha \in \Omega^1(S, \mathfrak{g}^*)$, set

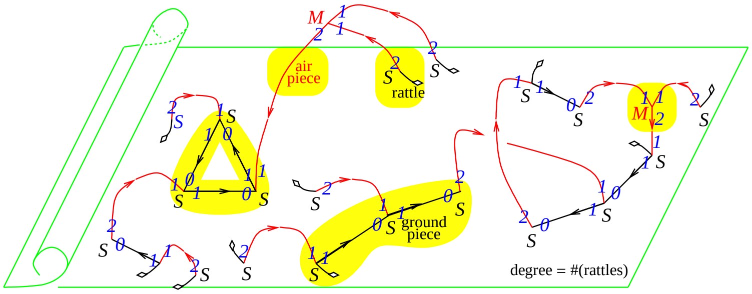
$$\mathcal{O}(A,B,\kappa) := \int \mathcal{D}\beta\mathcal{D}\alpha \exp\left(\frac{i}{\hbar}\int_S \langle \beta, d_{\kappa^*A}\alpha + \kappa^*B\rangle\right).$$

Cattaneo

The BF Feynman Rules. For an edge e, let Φ_e be its direction, in S^3 or S^1. Let ω_3 and ω_1 be volume forms on S^3 and S_1. Then

$$Z_{BF} = \sum_{\substack{\text{diagrams}\\ D}} \frac{[D]}{|\mathrm{Aut}(D)|} \underbrace{\int_{\mathbb{R}^2}\cdots\int_{\mathbb{R}^2}}_{S\text{-vertices}} \underbrace{\int_{\mathbb{R}^4}\cdots\int_{\mathbb{R}^4}}_{M\text{-vertices}} \prod_{\substack{\text{red}\\ e\in D}} \Phi_e^*\omega_3 \prod_{\substack{\text{black}\\ e\in D}} \Phi_e^*\omega_1$$

(modulo some STU- and IHX-like relations).



Issues. • Signs don't quite work out, and BF seems to reproduce only "half" of the wheels invariant.

• There are many more configuration space integrals than BF Feynman diagrams and than just trees and wheels.

• I don't know how to define "finite type" for arbitrary 2-knots.

> "God created the knots, all else in topology is the work of mortals."
>
> Leopold Kronecker (modified)

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Meta−Groups, Meta−Bicrossed−Products, and the Alexander Polynomial, 1

Dror Bar−Natan in Montreal, June 2013.

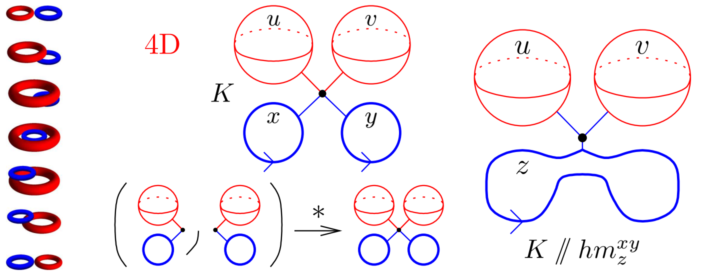
http://www.math.toronto.edu/~drorbn/Talks/Montreal-1306/

Abstract. I will define "meta-groups" and explain how one specific meta-group, which in itself is a "meta-bicrossed-product", gives rise to an "ultimate Alexander invariant" of tangles, that contains the Alexander polynomial (multivariable, if you wish), has extremely good composition properties, is evaluated in a topologically meaningful way, and is least-wasteful in a computational sense. If you believe in categorification, that's a wonderful playground.

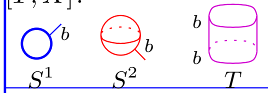
This work is closely related to work by Le Dimet (Comment. Math. Helv. **67** (1992) 306–315), Kirk, Livingston and Wang (arXiv:math/9806035) and Cimasoni and Turaev (arXiv:math.GT/0406269).

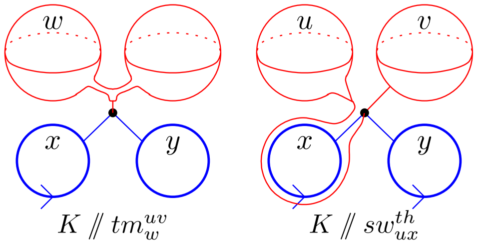
See also Dror Bar-Natan and Sam Selmani, *Meta-Monoids, Meta-Bicrossed Products, and the Alexander Polynomial*, arXiv:1302.5689.

Sam Selmani



Riddle. People often study $\pi_1(X)=[S^1,X]$ and $\pi_2(X)=[S^2,X]$. Why not $\pi_T(X):=[T,X]$?





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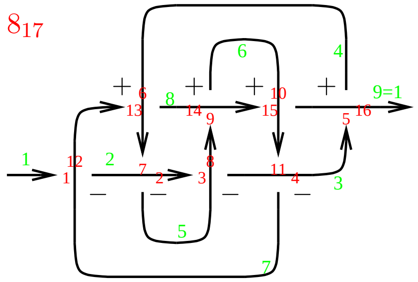
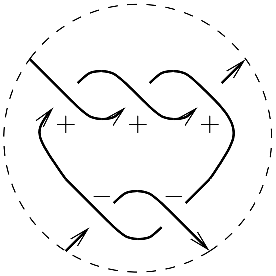
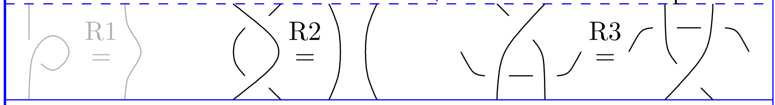


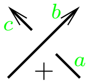


"divide and conquer"



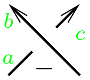
A Standard Alexander Formula.

Label the arcs 1 through $(n+1)=1$, make an $n\times n$ matrix as below, delete one row and one column, and compute the determinant:



$\longrightarrow$

	a	b	c
c	-1	$1-X$	X



$\longrightarrow$

	a	b	c
c	$-X$	$X-1$	1

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & x-1 & 0 & -x \\ -1 & x & 0 & 0 & 0 & 0 & 1-x & 0 \\ 0 & -1 & x & 0 & 1-x & 0 & 0 & 0 \\ x-1 & 0 & -x & 1 & 0 & 0 & 0 & 0 \\ 0 & 1-x & 0 & -1 & x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -x & 1 & 0 & x-1 \\ 0 & 0 & 1-x & 0 & 0 & -1 & x & 0 \\ 0 & 0 & 0 & x-1 & 0 & 0 & -x & 1 \end{pmatrix}$$

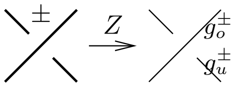
```
[[1 ;; 7, 1 ;; 7]] // Det
```

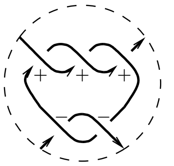
```
-1 + 4 X - 8 X^2 + 11 X^3 - 8 X^4 + 4 X^5 - X^6
```

Alexander Issues.

- Quick to compute, but computation departs from topology.
- Extends to tangles, but at an exponential cost.
- Hard to categorify.

Idea. Given a group G and two "YB" pairs $R^\pm=(g^\pm_o,g^\pm_u)\in G^2$, map them to xings and "multiply along", so that

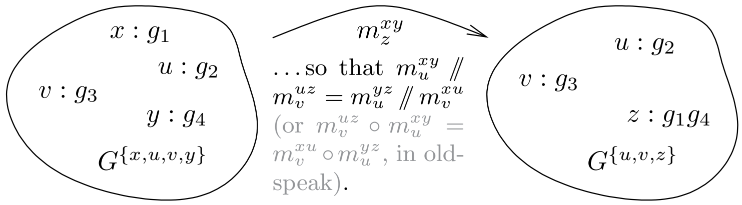




$$\xmapsto{Z}\ \begin{pmatrix} g^+_o g^+_u g^+_o g^-_u g^-_o g^+_u g^+_o g^+_u \\ g^-_u g^-_o \end{pmatrix}$$

This Fails! R2 implies that $g^\pm_o g^\mp_o = e = g^\pm_u g^\mp_u$ and then R3 implies that g^+_o and g^+_u commute, so the result is a simple counting invariant.

A Group Computer. Given G, can store group elements and perform operations on them:



…so that $m^{xy}_u\,/\!\!/\,m^{uz}_v=m^{yz}_u\,/\!\!/\,m^{xu}_v$ (or $m^{uz}_v\circ m^{xy}_u=m^{xu}_v\circ m^{yz}_u$, in old-speak).

Also has S_x for inversion, e_x for unit insertion, d_x for register deletion, Δ^z_{xy} for element cloning, ρ^x_y for renamings, and $(D_1,D_2)\mapsto D_1\cup D_2$ for merging, and many obvious composition axioms relating those. $P=\{x:g_1,y:g_2\}\Rightarrow P=\{d_yP\}\cup\{d_xP\}$

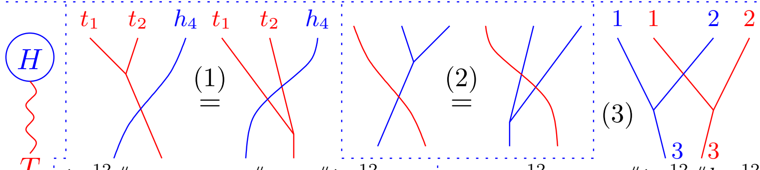
A Meta-Group. Is a similar "computer", only its internal structure is unknown to us. Namely it is a collection of sets $\{G_\gamma\}$ indexed by all finite sets γ, and a collection of operations m^{xy}_z, S_x, e_x, d_x, Δ^z_{xy} (sometimes), ρ^x_y, and $\cup$, satisfying the exact same *linear* properties.

Example 0. The non-meta example, $G_\gamma:=G^\gamma$.

Example 1. $G_\gamma:=M_{\gamma\times\gamma}(\mathbb{Z})$, with simultaneous row and column operations, and "block diagonal" merges. Here if $P=\begin{pmatrix} x: & a & b \\ y: & c & d\end{pmatrix}$ then $d_yP=(x:a)$ and $d_xP=(y:d)$ so $\{d_yP\}\cup\{d_xP\}=\begin{pmatrix} x: & a & 0 \\ y: & 0 & d\end{pmatrix}\neq P$. So this G is truly meta.

Claim. From a meta-group G and YB elements $R^\pm\in G_2$ we can construct a knot/tangle invariant.

Bicrossed Products. If $G=HT$ is a group presented as a product of two of its subgroups, with $H\cap T=\{e\}$, then also $G=TH$ and G is determined by H, T, and the "swap" map $sw^{th}:(t,h)\mapsto(h',t')$ defined by $th=h't'$. The map sw satisfies (1) and (2) below; conversely, if $sw:T\times H\to H\times T$ satisfies (1) and (2) (+ lesser conditions), then (3) defines a group structure on $H\times T$, the "bicrossed product".



$tm^{12}_1\,/\!\!/\,sw_{14}=sw_{24}\,/\!\!/\,sw_{14}\,/\!\!/\,tm^{12}_1$

$gm^{12}_3:=sw_{12}\,/\!\!/\,tm^{12}_3\,/\!\!/\,hm^{12}_3$

$h_1t_1h_2t_2=h_1(h'_2t'_1)t_2=(h_1h'_2)(t'_1t_2)=h_3t_3$

Meta−Groups, Meta−Bicrossed−Products, and the Alexander Polynomial, 2

A **Meta-Bicrossed-Product** is a collection of sets $\beta(\eta,\tau)$ and operations tm^{uv}_w, hm^{xy}_z and sw^{th}_{ux} (and lesser ones), such that tm and hm are "associative" and (1) and (2) hold (+ lesser conditions). A meta-bicrossed-product defines a meta-group with $G_\gamma := \beta(\gamma,\gamma)$ and gm as in (3).

Example. Take $\beta(\eta,\tau) = M_{\tau\times\eta}(\mathbb{Z})$ with row operations for the tails, column operations for the heads, and a trivial swap.

β Calculus. Let $\beta(\eta,\tau)$ be

$$\left\{\begin{array}{c|ccc} \omega & h_1 & h_2 & \cdots \\ \hline t_1 & \alpha_{11} & \alpha_{12} & \cdot \\ t_2 & \alpha_{21} & \alpha_{22} & \cdot \\ \vdots & . & . & . \end{array}\ \middle|\ \begin{array}{l} h_j \in \eta,\ t_i \in \tau, \text{ and } \omega \text{ and} \\ \text{the } \alpha_{ij} \text{ are rational func-} \\ \text{tions in a variable } X \text{ with} \\ \omega(1) = 1 \text{ and } \alpha_{ij}(1) = 0 \end{array}\right\},$$

$$tm^{uv}_w : \begin{array}{c|c} \omega & \cdots \\ \hline t_u & \alpha \\ t_v & \beta \\ \vdots & \gamma \end{array} \mapsto \begin{array}{c|c} \omega & \cdots \\ \hline t_w & \alpha+\beta \\ \vdots & \gamma \end{array}, \qquad \begin{array}{c|c} \omega_1 & \eta_1 \\ \hline \tau_1 & \alpha_1 \end{array} \cup \begin{array}{c|c} \omega_2 & \eta_2 \\ \hline \tau_2 & \alpha_2 \end{array} = \begin{array}{c|cc} \omega_1\omega_2 & \eta_1 & \eta_2 \\ \hline \tau_1 & \alpha_1 & 0 \\ \tau_2 & 0 & \alpha_2 \end{array},$$

$$hm^{xy}_z : \begin{array}{c|ccc} \omega & h_x & h_y & \cdots \\ \hline \vdots & \alpha & \beta & \gamma \end{array} \mapsto \begin{array}{c|cc} \omega & h_z & \cdots \\ \hline \vdots & \alpha+\beta+\langle\alpha\rangle\beta & \gamma \end{array},$$

$$sw^{th}_{ux} : \begin{array}{c|cc} \omega & h_x & \cdots \\ \hline t_u & \alpha & \beta \\ \vdots & \gamma & \delta \end{array} \mapsto \begin{array}{c|cc} \omega\epsilon & h_x & \cdots \\ \hline t_u & \alpha(1+\langle\gamma\rangle/\epsilon) & \beta(1+\langle\gamma\rangle/\epsilon) \\ \vdots & \gamma/\epsilon & \delta-\gamma\beta/\epsilon \end{array},$$

where $\epsilon := 1+\alpha$ and $\langle c\rangle := \sum_i c_i$, and let

$$R^p_{ab} := \begin{array}{c|cc} 1 & h_a & h_b \\ \hline t_a & 0 & X-1 \\ t_b & 0 & 0 \end{array} \qquad R^m_{ab} := \begin{array}{c|cc} 1 & h_a & h_b \\ \hline t_a & 0 & X^{-1}-1 \\ t_b & 0 & 0 \end{array}.$$

Theorem. Z^β is a tangle invariant (and more). Restricted to knots, the ω part is the Alexander polynomial. On braids, it is equivalent to the Burau representation. A variant for links contains the multivariable Alexander polynomial.

Why Happy? • Applications to w-knots.

- Everything that I know about the Alexander polynomial can be expressed cleanly in this language (even if without proof), except HF, but including genus, ribbonness, cabling, v-knots, knotted graphs, etc., and there's potential for vast generalizations.
- The least wasteful "Alexander for tangles" I'm aware of.
- Every step along the computation is the invariant of something.
- Fits on one sheet, including implementation & propaganda.

Further meta-monoids. Π (and variants), $\mathcal{A}$ (and quotients), vT, …

Further meta-bicrossed-products. Π (and variants), $\overrightarrow{\mathcal{A}}$ (and quotients), M_0, M, $\mathcal{K}^{bh}$, $\mathcal{K}^{rbh}$, …

Meta-Lie-algebras. $\mathcal{A}$ (and quotients), $\mathcal{S}$, …

Meta-Lie-bialgebras. $\overrightarrow{\mathcal{A}}$ (and quotients), …

I don't understand the relationship between gr and H, as it appears, for example, in braid theory.

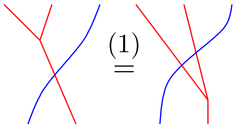
I mean business!

```
βSimp = Factor; SetAttributes[βCollect, Listable];
βCollect[B[ω_, Λ_]] := B[βSimp[ω],
   Collect[Λ, h_, Collect[#, t_, βSimp] &]];
βForm[B[ω_, Λ_]] := Module[{ts, hs, M},
   ts = Union[Cases[B[ω, Λ], t_u_ :→ u, Infinity]];
   hs = Union[Cases[B[ω, Λ], h_x_ :→ x, Infinity]];
   M = Outer[βSimp[Coefficient[Λ, h_#1 t_#2]] &, hs, ts];
   PrependTo[M, t_# & /@ ts];
   M = Prepend[Transpose[M], Prepend[h_# & /@ hs, ω]];
   MatrixForm[M]];
βForm[else_] := else /. β_B :→ βForm[β];
Format[β_B, StandardForm] := βForm[β];
```

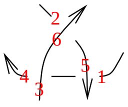
```
⟨μ_⟩ := μ /. t_ → 1;
tm_{u_ v_→w_}[β_] := βCollect[β /. t_{u|v} → t_w];
hm_{x_ y_→z_}[B[ω_, Λ_]] := Module[
   {α = D[Λ, h_x], β = D[Λ, h_y], γ = Λ /. h_{x|y} → 0},
   B[ω, (α + (1 + ⟨α⟩) β) h_z + γ] // βCollect];
sw_{u_ x_}[B[ω_, Λ_]] := Module[{α, β, γ, δ, ϵ},
   α = Coefficient[Λ, h_x t_u]; β = D[Λ, t_u] /. h_x → 0;
   γ = D[Λ, h_x] /. t_u → 0;    δ = Λ /. h_x | t_u → 0;
   ϵ = 1 + α;
   B[ω * ϵ, α (1 + ⟨γ⟩ / ϵ) h_x t_u + β (1 + ⟨γ⟩ / ϵ) t_u
        + γ / ϵ h_x                + δ - γ * β / ϵ
    ] // βCollect];
gm_{a_ b_→c_}[β_] := β // sw_{ab} // hm_{ab→c} // tm_{ab→c};
B /: B[ω1_, Λ1_] B[ω2_, Λ2_] := B[ω1 * ω2, Λ1 + Λ2];
Rp_{a_ b_} := B[1, (X - 1) t_a h_b];
Rm_{a_ b_} := B[1, (X^-1 - 1) t_a h_b];
```

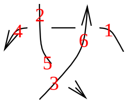
```
{β = B[ω, Sum[α_{10 i+j} t_i h_j, {i, {1, 2, 3}}, {j, {4, 5}}]],
  (β // tm_{12→1} // sw_{14}) == (β // sw_{24} // sw_{14} // tm_{12→1})}
```

$$\left\{\begin{pmatrix} \omega & h_4 & h_5 \\ t_1 & \alpha_{14} & \alpha_{15} \\ t_2 & \alpha_{24} & \alpha_{25} \\ t_3 & \alpha_{34} & \alpha_{35} \end{pmatrix}, \text{True}\right\}$$



Some testing





```
{Rm_{51} Rm_{62} Rp_{34} // gm_{14→1} // gm_{25→2} // gm_{36→3},
  Rp_{61} Rm_{24} Rm_{35} // gm_{14→1} // gm_{25→2} // gm_{36→3}}
```

$$\left\{\begin{pmatrix} 1 & h_1 & h_2 \\ t_2 & -\frac{-1+X}{X} & 0 \\ t_3 & \frac{-1+X}{X} & -\frac{-1+X}{X} \end{pmatrix}, \begin{pmatrix} 1 & h_1 & h_2 \\ t_2 & -\frac{-1+X}{X} & 0 \\ t_3 & \frac{-1+X}{X} & -\frac{-1+X}{X} \end{pmatrix}\right\}$$

… divide and conquer!

```
β = Rm_{12,1} Rm_{27} Rm_{83} Rm_{4,11} Rp_{16,5} Rp_{6,13} Rp_{14,9} Rp_{10,15}
```

8_{17}

$$\begin{pmatrix} 1 & h_1 & h_3 & h_5 & h_7 & h_9 & h_{11} & h_{13} & h_{15} \\ t_2 & 0 & 0 & 0 & -\frac{-1+X}{X} & 0 & 0 & 0 & 0 \\ t_4 & 0 & 0 & 0 & 0 & 0 & -\frac{-1+X}{X} & 0 & 0 \\ t_6 & 0 & 0 & 0 & 0 & 0 & 0 & -1+X & 0 \\ t_8 & 0 & -\frac{-1+X}{X} & 0 & 0 & 0 & 0 & 0 & 0 \\ t_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1+X \\ t_{12} & -\frac{-1+X}{X} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ t_{14} & 0 & 0 & 0 & 0 & -1+X & 0 & 0 & 0 \\ t_{16} & 0 & 0 & -1+X & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

```
Do[β = β // gm_{1k→1}, {k, 2, 10}]; β
```

8_{17}, cont.

$$\begin{pmatrix} \frac{1}{X} & h_1 & h_{11} & h_{13} & h_{15} \\ t_1 & -\frac{(-1+X)(1+X)}{X} & -(-1+X)(1-X+X^2) & (-1+X)(1-X+X^2) & -1+X \\ t_{12} & -\frac{-1+X}{X} & 0 & 0 & 0 \\ t_{14} & -1+X & \frac{(-1+X)^2(1-X+X^2)}{X} & -\frac{(-1+X)^2(1-X+X^2)}{X} & 0 \\ t_{16} & \frac{-1+X}{X} & (-1+X)^2 & -\frac{(-1+X)^3}{X} & 0 \end{pmatrix}$$

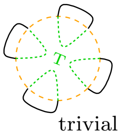
James
Waddell
Alexander

```
Do[β = β // gm_{1k→1}, {k, 11, 16}]; β
```

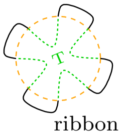
$$\left(-\frac{1-4X+8X^2-11X^3+8X^4-4X^5+X^6}{X^3}\right)$$

A Partial To Do List. 1. Where does it *more simply* come from?

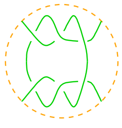
2. Remove all the denominators.
3. How do determinants arise in this context?
4. Understand links ("meta-conjugacy classes").
5. Find the "reality condition".
6. Do some "Algebraic Knot Theory".
7. Categorify.
8. Do the same in other natural quotients of the v/w-story.



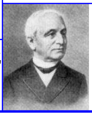
trivial



ribbon



example



"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

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Trees and Wheels and Balloons and Hoops

Dror Bar-Natan, Zurich, September 2013

`ωεβ:=http://www.math.toronto.edu/~drorbn/Talks/Zurich-130919`

15 Minutes on Algebra

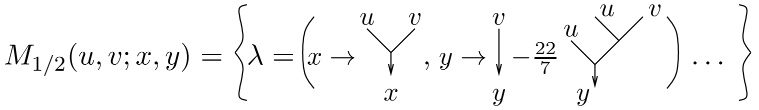
Let T be a finite set of "tail labels" and H a finite set of "head labels". Set

$$M_{1/2}(T;H) := FL(T)^H,$$

"H-labeled lists of elements of the degree-completed free Lie algebra generated by T".

$$FL(T) = \left\{2t_2 - \frac{1}{2}[t_1,[t_1,t_2]] + \dots\right\} \Big/ \begin{pmatrix}\text{anti-symmetry}\\ \text{Jacobi}\end{pmatrix}$$

… with the obvious bracket.



Operations $M_{1/2} \to M_{1/2}$. newspeak!

Tail Multiply tm^{uv}_w is $\lambda \mapsto \lambda \sslash (u,v \to w)$, satisfies "meta-associativity", $tm^{uv}_u \sslash tm^{uw}_u = tm^{vw}_v \sslash tm^{uv}_u$.

Head Multiply hm^{xy}_z is $\lambda \mapsto (\lambda \backslash \{x,y\}) \cup (z \to \mathrm{bch}(\lambda_x,\lambda_y))$, where

$$\mathrm{bch}(\alpha,\beta) := \log(e^\alpha e^\beta) = \alpha + \beta + \frac{[\alpha,\beta]}{2} + \frac{[\alpha,[\alpha,\beta]]+[[\alpha,\beta],\beta]}{12} + \dots$$

satisfies $\mathrm{bch}(\mathrm{bch}(\alpha,\beta),\gamma) = \log(e^\alpha e^\beta e^\gamma) = \mathrm{bch}(\alpha,\mathrm{bch}(\beta,\gamma))$ and hence meta-associativity, $hm^{xy}_x \sslash hm^{xz}_x = hm^{yz}_y \sslash hm^{xy}_x$.

Tail by Head Action tha^{ux} is $\lambda \mapsto \lambda \sslash RC^{\lambda_x}_u$, where $C^{-\gamma}_u\colon FL \to FL$ is the substitution $u \to e^{-\gamma} u e^{\gamma}$, or more precisely,

$$C^{-\gamma}_u\colon u \to e^{-\operatorname{ad}\gamma}(u) = u - [\gamma,u] + \frac{1}{2}[\gamma,[\gamma,u]] - \dots,$$

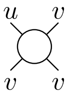
and $RC^\gamma_u = (C^{-\gamma}_u)^{-1}$. Then $C^{\mathrm{bch}(\alpha,\beta)}_u = C^{\alpha \sslash RC^{-\beta}_u}_u \sslash C^\beta_u$ hence $RC^{\mathrm{bch}(\alpha,\beta)}_u = RC^\alpha_u \sslash RC^{\beta \sslash RC^\alpha_u}_u$ hence "meta $u^{xy} = (u^x)^y$",

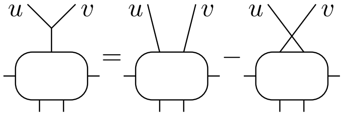
$$hm^{xy}_z \sslash tha^{uz} = tha^{ux} \sslash tha^{uy} \sslash hm^{xy}_z,$$

and $tm^{uv}_w \sslash C^{\gamma \sslash tm^{uv}_w}_w = C^{\gamma \sslash RC^{-\gamma}_v}_u \sslash C^\gamma_v \sslash tm^{uv}_w$ and hence "meta $(uv)^x = u^x v^x$", $tm^{uv}_w \sslash tha^{wx} = tha^{ux} \sslash tha^{vx} \sslash tm^{uv}_w$.

Wheels. Let $M(T;H) := M_{1/2}(T;H) \times CW(T)$, where $CW(T)$ is the (completed graded) vector space of cyclic words on T, or equaly well, on $FL(T)$:


ωεβ/antiq-ave

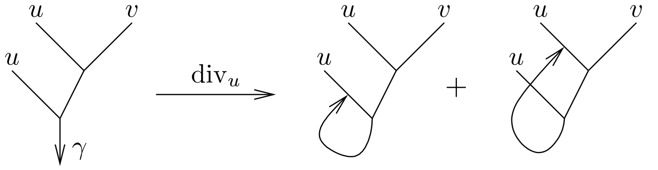




Operations. On $M(T;H)$, define tm^{uv}_w and hm^{xy}_z as before, and tha^{ux} by adding some J-spice:

$$(\lambda;\omega) \mapsto (\lambda, \omega + J_u(\lambda_x)) \sslash RC^{\lambda_x}_u,$$

where $J_u(\gamma) := \int_0^1 ds\, \mathrm{div}_u(\gamma \sslash RC^{s\gamma}_u) \sslash C^{-s\gamma}_u$, and




Alekseev

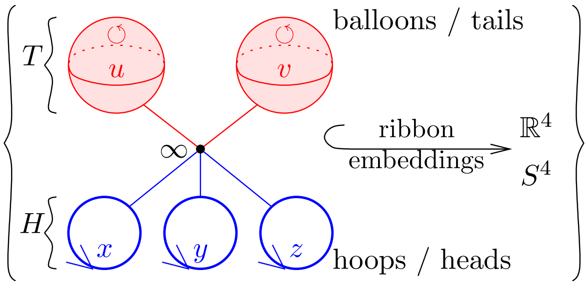

Torossian

Theorem Blue. All blue identities still hold.

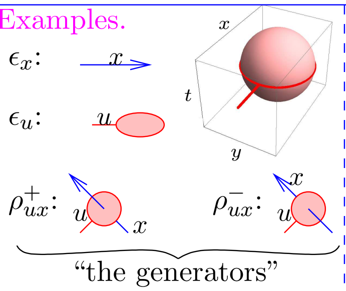
Merge Operation. $(\lambda_1;\omega_1) * (\lambda_2;\omega_2) := (\lambda_1 \cup \lambda_2; \omega_1 + \omega_2)$.

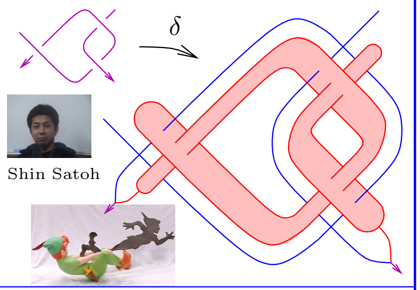
15 Minutes on Topology

$\mathcal{K}^{bh}(T;H)$. "Ribbon-knotted balloons and hoops"

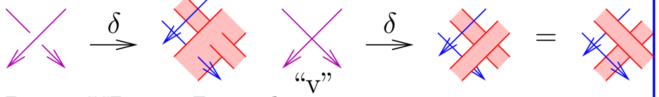


Examples.

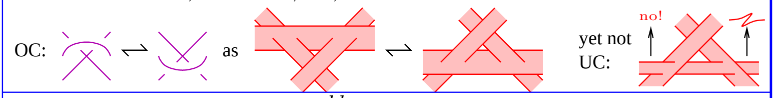




More on δ



satisfies R123, VR123, D, and

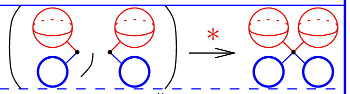


- δ injects u-knots into $\mathcal{K}^{bh}$ (likely u-tangles too).
- δ maps v-tangles to $\mathcal{K}^{bh}$; the kernel contains the above and conjecturally (Satoh), that's all.
- Allowing punctures and cuts, δ is onto.

Operations

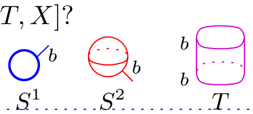
Punctures & Cuts

Connected Sums.

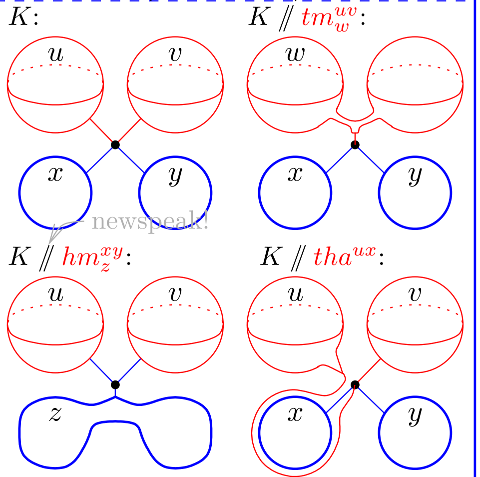


If X is a space, $\pi_1(X)$ is a group, $\pi_2(X)$ is an Abelian group, and π_1 acts on π_2.

Riddle. People often study $\pi_1(X) = [S^1,X]$ and $\pi_2(X) = [S^2,X]$. Why not $\pi_T(X) := [T,X]$?



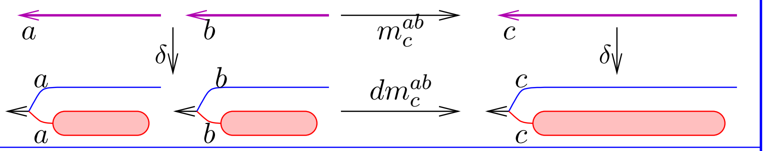
"Meta-Group-Action"



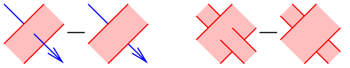
Properties.

- Associativities: $m^{ab}_a \sslash m^{ac}_a = m^{bc}_b \sslash m^{ab}_a$, for $m = tm, hm$.
- "$(uv)^x = u^x v^x$": $tm^{uv}_w \sslash tha^{wx} = tha^{ux} \sslash tha^{vx} \sslash tm^{uv}_w$,
- "$u^{(xy)} = (u^x)^y$": $hm^{xy}_z \sslash tha^{uz} = tha^{ux} \sslash tha^{uy} \sslash hm^{xy}_z$.

Tangle concatenations → $\pi_1 \ltimes \pi_2$. With $dm^{ab}_c := tha^{ab} \sslash tm^{ab}_c \sslash hm^{ab}_c$,



Finite type invariants make sense in the usual way, and "algebra" is (the primitive part of) "gr" of "topology".

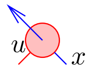


Trees and Wheels and Balloons and Hoops: Why I Care

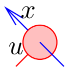
Moral. To construct an M-valued invariant ζ of (v-)tangles, and nearly an invariant on $\mathcal{K}^{bh}$, it is enough to declare ζ on the generators, and verify the relations that δ satisfies.

The Invariant ζ. Set $\zeta(\epsilon_x) = (x \to 0; 0)$, $\zeta(\epsilon_u) = ((); 0)$, and

ζ:



$$\longmapsto \left(\begin{array}{c} u \\ \downarrow \\ x \end{array} ; 0 \right)$$

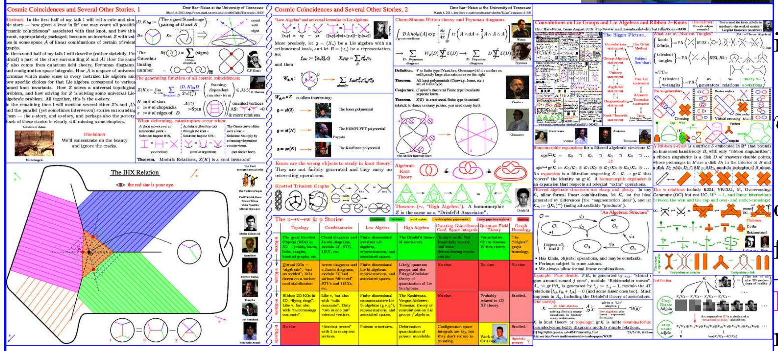


$$\longmapsto \left(- \begin{array}{c} u \\ \downarrow \\ x \end{array} ; 0 \right)$$

Theorem. ζ is (log of) the unique homomorphic universal finite type invariant on $\mathcal{K}^{bh}$.

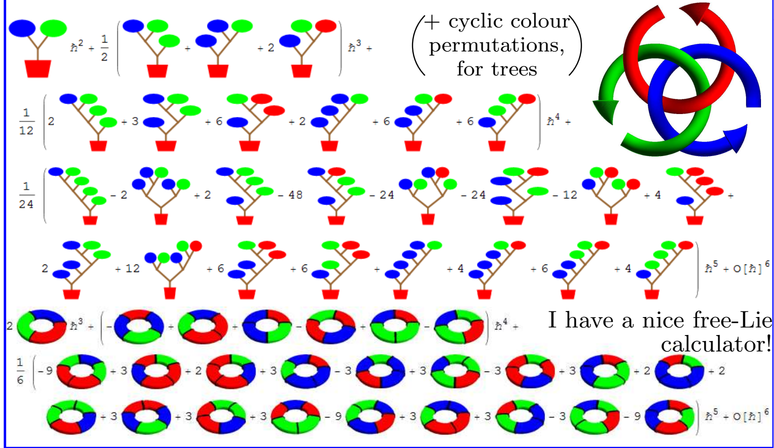
(… and is the tip of an iceberg)

Paper in progress with Dancso, ωεβ/wko



See also ωεβ/tenn, ωεβ/bonn, ωεβ/swiss, ωεβ/portfolio

ζ is computable! ζ of the Borromean tangle, to degree 5:



(+ cyclic colour permutations, for trees)

I have a nice free-Lie calculator!

Tensorial Interpretation. Let $\mathfrak{g}$ be a finite dimensional Lie algebra (any!). Then there's $\tau : FL(T) \to \mathrm{Fun}(\oplus_T\mathfrak{g} \to \mathfrak{g})$ and $\tau : CW(T) \to \mathrm{Fun}(\oplus_T\mathfrak{g})$. Together, $\tau : M(T;H) \to \mathrm{Fun}(\oplus_T\mathfrak{g} \to \oplus_H\mathfrak{g})$, and hence

$$e^\tau : M(T;H) \to \mathrm{Fun}(\oplus_T\mathfrak{g} \to \mathcal{U}^{\otimes H}(\mathfrak{g})).$$

ζ and BF Theory. (See Cattaneo-Rossi, arXiv:math-ph/0210037) Let A denote a $\mathfrak{g}$-connection on S^4 with curvature F_A, and B a $\mathfrak{g}^*$-valued 2-form on S^4. For a hoop γ_x, let $\mathrm{hol}_{\gamma_x}(A) \in \mathcal{U}(\mathfrak{g})$ be the holonomy of A along γ_x. For a ball γ_u, let $\mathcal{O}_{\gamma_u}(B) \in \mathfrak{g}^*$ be (roughly) the integral of B (transported via A to ∞) on γ_u.

Cattaneo

Loose Conjecture. For $\gamma \in \mathcal{K}(T;H)$,

$$\int \mathcal{D}A\mathcal{D}B e^{\int B\wedge F_A} \prod_u e^{\mathcal{O}_{\gamma_u}(B)} \bigotimes_x \mathrm{hol}_{\gamma_x}(A) = e^\tau(\zeta(\gamma)).$$

That is, ζ is a complete evaluation of the BF TQFT.

> “God created the knots, all else in topology is the work of mortals.”
>
> Leopold Kronecker (modified)

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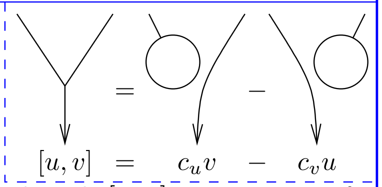
The Knot Atlas
Anyone Can Edit

The β quotient is M divided by all relations that universally hold when when $\mathfrak{g}$ is the 2D non-Abelian Lie algebra. Let $R = \mathbb{Q}[\![\{c_u\}_{u\in T}]\!]$ and $L_\beta := R \otimes T$ with central R and with $[u,v] = c_uv - c_vu$ for $u, v \in T$. Then $FL \to L_\beta$ and $CW \to R$. Under this,



$$[u,v] \quad = \quad c_uv \quad - \quad c_vu$$

$$\mu \to ((\lambda_x); \omega) \quad \text{with } \lambda_x = \sum_{u\in T} \lambda_{ux}ux, \quad \lambda_{ux}, \omega \in R,$$

$$\mathrm{bch}(u,v) \to \frac{c_u + c_v}{e^{c_u+c_v} - 1}\left(\frac{e^{c_u} - 1}{c_u}u + e^{c_u}\frac{e^{c_v} - 1}{c_v}v\right),$$

if $\gamma = \sum \gamma_v v$ then with $c_\gamma := \sum \gamma_v c_v$,

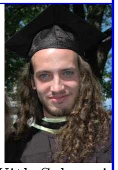
$$u /\!\!/ RC_u^\gamma = \left(1 + c_u\gamma_u\frac{e^{c_\gamma} - 1}{c_\gamma}\right)^{-1}\left(e^{c_\gamma}u - c_u\frac{e^{c_\gamma} - 1}{c_\gamma}\sum_{v\neq u}\gamma_v v\right),$$

$\mathrm{div}_u\,\gamma = c_u\gamma_u$, and $J_u(\gamma) = \log\left(1 + \frac{e^{c_\gamma}-1}{c_\gamma}c_u\gamma_u\right)$, so ζ is formula-computable to all orders! Can we simplify?

Repackaging. Given $((x \to \lambda_{ux}); \omega)$, set $c_x := \sum_v c_v\lambda_{vx}$, replace $\lambda_{ux} \to \alpha_{ux} := c_u\lambda_{ux}\frac{e^{c_x}-1}{c_x}$ and $\omega \to e^\omega$, use $t_u = e^{c_u}$, and write α_{ux} as a matrix. Get “β calculus”.

β Calculus. Let $\beta(T;H)$ be

$$\left\{ \begin{array}{c|ccc} \omega & x & y & \cdots \\ \hline u & \alpha_{ux} & \alpha_{uy} & \cdot \\ v & \alpha_{vx} & \alpha_{vy} & \cdot \\ \vdots & \cdot & \cdot & \cdot \end{array} \middle| \ \begin{array}{l} \omega \text{ and the } \alpha_{ux}\text{'s are} \\ \text{rational functions in} \\ \text{variables } t_u \text{, one for} \\ \text{each } u \in T. \end{array} \right\},$$



With Selmani, ωεβ/meta

$$tm^{uv}_w : \begin{array}{c|c} \omega & \cdots \\ \hline u & \alpha \\ v & \beta \\ \vdots & \gamma \end{array} \mapsto \begin{array}{c|c} \omega & \cdots \\ \hline w & \alpha+\beta \\ \vdots & \gamma \end{array}, \qquad \begin{array}{c|c} \omega_1 & H_1 \\ \hline T_1 & \alpha_1 \end{array} * \begin{array}{c|c} \omega_2 & H_2 \\ \hline T_2 & \alpha_2 \end{array} = \begin{array}{c|cc} \omega_1\omega_2 & H_1 & H_2 \\ \hline T_1 & \alpha_1 & 0 \\ T_2 & 0 & \alpha_2 \end{array},$$

$$hm^{xy}_z : \begin{array}{c|ccc} \omega & x & y & \cdots \\ \hline \vdots & \alpha & \beta & \gamma \end{array} \mapsto \begin{array}{c|cc} \omega & z & \cdots \\ \hline \vdots & \alpha+\beta+\langle\alpha\rangle\beta & \gamma \end{array},$$

$$tha^{ux} : \begin{array}{c|cc} \omega & x & \cdots \\ \hline u & \alpha & \beta \\ \vdots & \gamma & \delta \end{array} \mapsto \begin{array}{c|cc} \omega\epsilon & x & \cdots \\ \hline u & \alpha(1+\langle\gamma\rangle/\epsilon) & \beta(1+\langle\gamma\rangle/\epsilon) \\ \vdots & \gamma/\epsilon & \delta - \gamma\beta/\epsilon \end{array},$$

where $\epsilon := 1 + \alpha$, $\langle\alpha\rangle := \sum_v \alpha_v$, and $\langle\gamma\rangle := \sum_{v\neq u}\gamma_v$, and let

$$R^+_{ux} := \begin{array}{c|c} 1 & x \\ \hline u & t_u - 1 \end{array} \qquad R^-_{ux} := \begin{array}{c|c} 1 & x \\ \hline u & t_u^{-1} - 1 \end{array}.$$

On long knots, ω is the Alexander polynomial!

Why happy? An ultimate Alexander invariant: Manifestly polynomial (time and size) extension of the (multivariable) Alexander polynomial to tangles. Every step of the computation is the computation of the invariant of some topological thing (no fishy Gaussian elimination). *If there should be an Alexander invariant with a computable algebraic categorification, it is this one!*

See also ωεβ/regina, ωεβ/caen, ωεβ/newton.

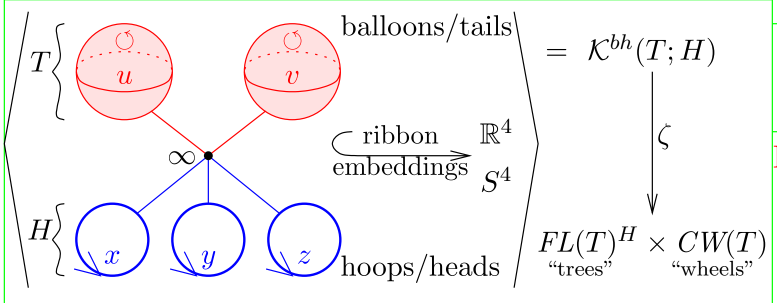
May class: ωεβ/aarhus **Class next year:** ωεβ/1350

Paper: ωεβ/kbh

Dror Bar-Natan: Talks: Geneva-131024: ω :=http://www.math.toronto.edu/~drorbn/Talks/Geneva-131024

Finite Type Invariants of Ribbon Knotted Balloons and Hoops

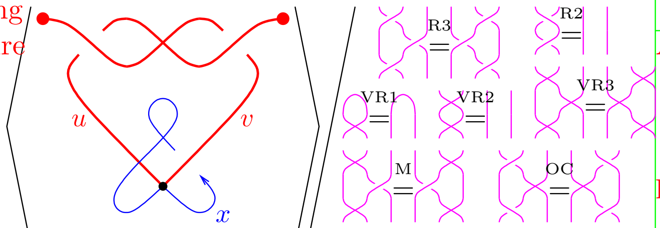
Abstract. On my September 17 Geneva talk (ω/sep) I described a certain trees-and-wheels-valued invariant ζ of ribbon knotted loops and 2-spheres in 4-space, and my October 8 Geneva talk (ω/oct) describes its reduction to the Alexander polynomial. Today I will explain how that same invariant arises completely naturally within the theory of finite type invariants of ribbon knotted loops and 2-spheres in 4-space.

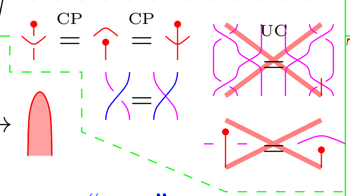


My goal is to tell you why such an invariant is expected, yet not to derive the computable formulas.

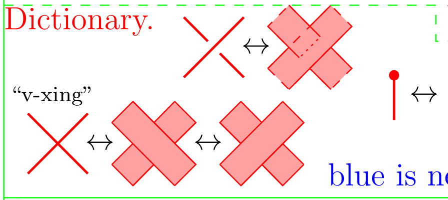
Disturbing Conjecture


Satoh





Dictionary.



blue is never "over"

Expansions

the semi-virtual

Let $\mathcal{I}^n := \langle$pictures with $\geq n$ semi-virts$\rangle \subset \mathcal{K}^{bh}$.
We seek an "expansion"

$$Z\colon \mathcal{K}^{bh} \to \operatorname{gr}\mathcal{K}^{bh} = \widehat{\bigoplus} \mathcal{I}^n/\mathcal{I}^{n+1} =: \mathcal{A}^{bh}$$

satisfying "property U": if $\gamma \in \mathcal{I}^n$, then

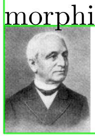
$$Z(\gamma) = (0, \dots, 0, \gamma/\mathcal{I}^{n+1}, *, *, \dots).$$


X.-S. Lin

Why? • Just because, and this is vastly more general.
- $\left(\mathcal{K}^{bh}/\mathcal{I}^{n+1}\right)^\star$ is "finite-type/polynomial invariants".
- The Taylor example: Take $\mathcal{K} = C^\infty(\mathbb{R}^n)$, $\mathcal{I} = \{f \in \mathcal{K}\colon f(0) = 0\}$. Then $\mathcal{I}^n = \{f\colon f$ vanishes like $|x|^n\}$ so $\mathcal{I}^n/\mathcal{I}^{n+1}$ is homogeneous polynomials of degree n and Z is a "Taylor expansion"! (So Taylor expansions are vastly more general than you'd think).

Plan. We'll construct a graded $\tilde{\mathcal{A}}^{bh}$, a surjective graded $\pi\colon \tilde{\mathcal{A}}^{bh} \to \mathcal{A}^{bh}$, and a filtered $\tilde{Z}\colon \mathcal{K}^{bh} \to \mathcal{A}^{bh}$ so that $\pi /\!\!/ \operatorname{gr}\tilde{Z} = Id$ (property U: if $\deg D = n$, $\tilde{Z}(\pi(D)) = \pi(D) + (\deg \geq n)$). Hence • π is an isomorphism. • $Z := \tilde{Z} /\!\!/ \pi$ is an expansion.

$$\begin{tikzcd} & \tilde{\mathcal{A}}^{bh} \arrow[d, shift right, "\pi"'] \\ \mathcal{K}^{bh} \arrow[ur, "\tilde{Z}"] \arrow[r, "Z"'] & \mathcal{A}^{bh} \arrow[u, shift right, "\operatorname{gr}\tilde{Z}"'] \end{tikzcd}$$



"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

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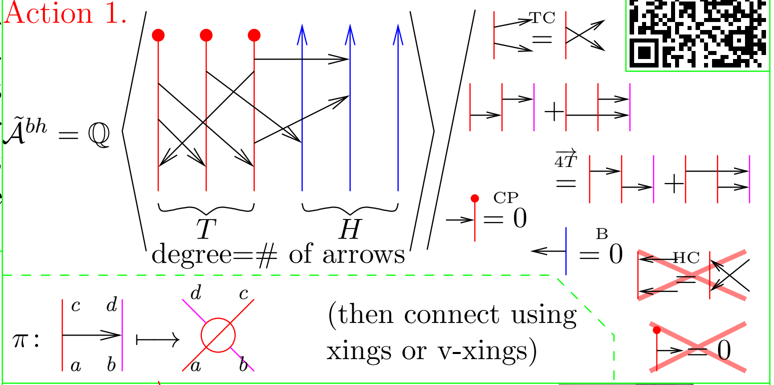
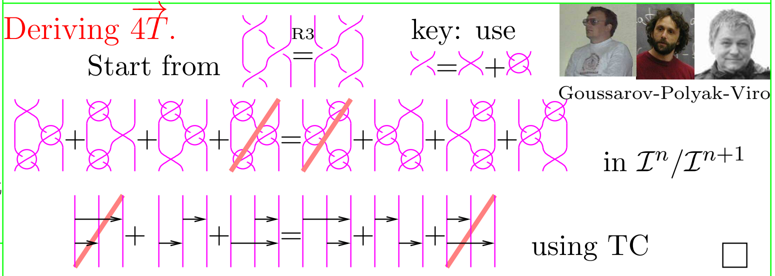
Action 1.



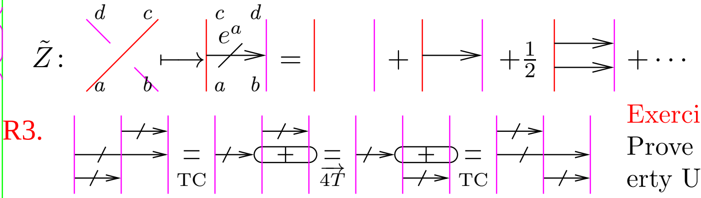
(then connect using xings or v-xings)

Deriving $\overrightarrow{4T}$.



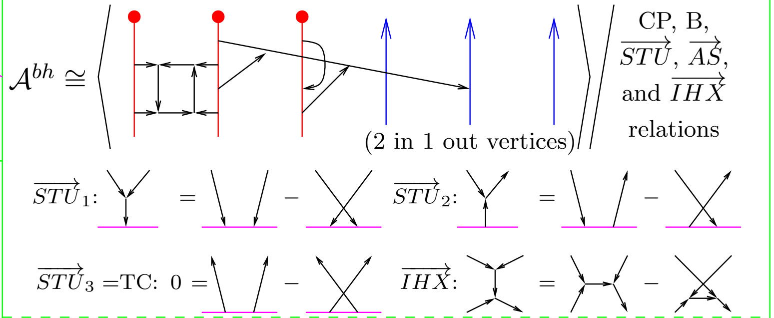
□

Action 2.



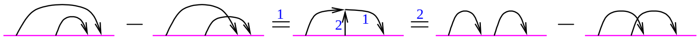
Exercise. Prove property U.

The Bracket-Rise Theorem.



CP, B, $\overrightarrow{STU}$, $\overrightarrow{AS}$, and $\overrightarrow{IHX}$ relations

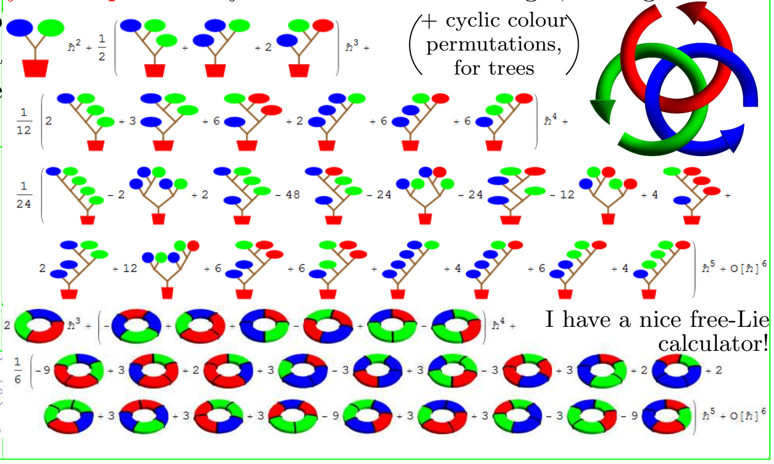
Proof.



Corollaries. (1) Related to Lie algebras! (2) Only trees and wheels persist.

Theorem. $\mathcal{A}^{bh}$ is a bi-algebra. The space of its primitives is $FL(T)^H \times CW(T)$, and $\zeta = \log Z$.

ζ is computable! ζ of the Borromean tangle, to degree 5:



I have a nice free-Lie calculator!

Dror Bar-Natan: Academic Pensieve: 2014-04: BF2C:
`http://drorbn.net/AcademicPensieve/2014-04/BF2C`
continues `http://www.math.toronto.edu/~drorbn/Talks/Vienna-1402`

A Partial Reduction of BF Theory to Combinatorics, 1

Abstract. I will describe a semi-rigorous reduction of perturbative BF theory (Cattaneo-Rossi [CR]) to computable combinatorics, in the case of ribbon 2-links. Also, I will explain how and why my approach may or may not work in the non-ribbon case. Weak this result is, and at least partially already known (Watanabe [Wa]). Yet in the ribbon case, the resulting invariant is a universal finite type invariant, a gadget that significantly generalizes and clarifies the Alexander polynomial and that is closely related to the Kashiwara-Vergne problem. I cannot rule out the possibility that the corresponding gadget in the non-ribbon case will be as interesting. (good news in highlight)

BF Following [CR]. $A \in \Omega^1(M = \mathbb{R}^4, \mathfrak{g})$, $B \in \Omega^2(M, \mathfrak{g}^*)$,

$$S(A,B) := \int_M \langle B, F_A \rangle.$$

With $\kappa\colon (S = \mathbb{R}^2) \to M$, $\beta \in \Omega^0(S, \mathfrak{g})$, $\alpha \in \Omega^1(S, \mathfrak{g}^*)$, set

$$\mathcal{O}(A,B,\kappa) := \int \mathcal{D}\beta\mathcal{D}\alpha \exp\left(\frac{i}{\hbar}\int_S \langle \beta, d_{\kappa^* A}\alpha + \kappa^* B\rangle\right).$$

The BF Feynman Rules. For an edge e, let Φ_e be its direction, in S^3 or S^1. Let ω_3 and ω_1 be volume forms on S^3 and S_1. Then for a 2-link $(\kappa_t)_{t\in T}$,

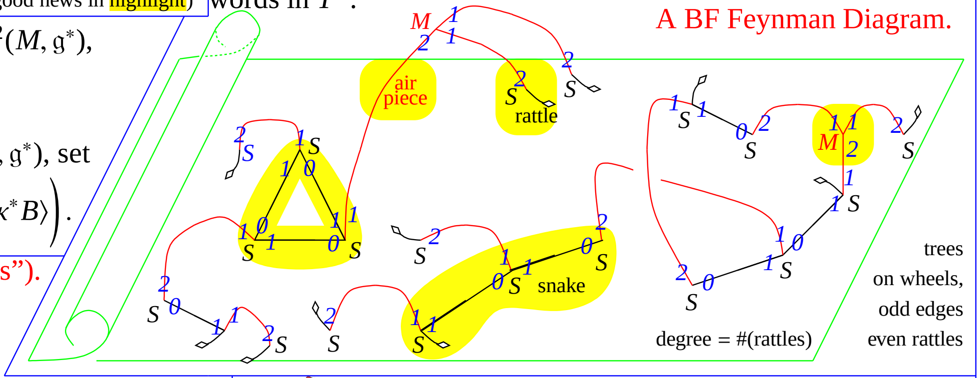

Cattaneo

Rossi

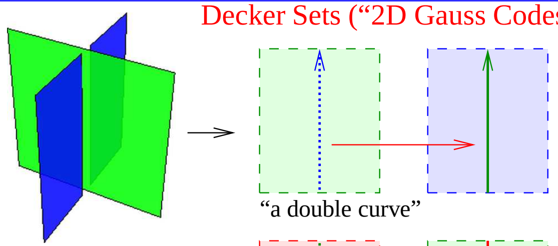
$$\zeta = \log \sum_{\substack{\text{diagrams}\\ D}} \frac{[D]}{|\mathrm{Aut}(D)|} \underbrace{\int_{\mathbb{R}^2}\cdots\int_{\mathbb{R}^2}}_{S\text{-vertices}} \underbrace{\int_{\mathbb{R}^4}\cdots\int_{\mathbb{R}^4}}_{M\text{-vertices}} \prod_{\substack{\text{red}\\ e\in D}} \Phi_e^*\omega_3 \prod_{\substack{\text{black}\\ e\in D}} \Phi_e^*\omega_1$$

is an invariant in $CW(FL(T)) \to CW(T)/\sim$, "symmetrized cyclic words in T".

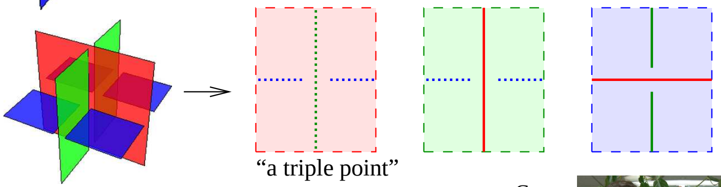
A BF Feynman Diagram.

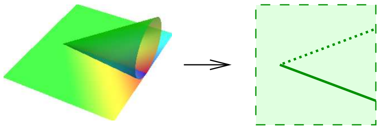


Decker Sets ("2D Gauss Codes").

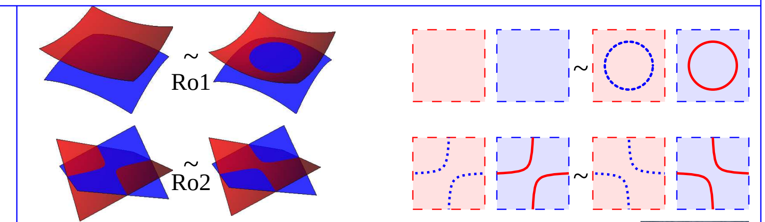

"a double curve"

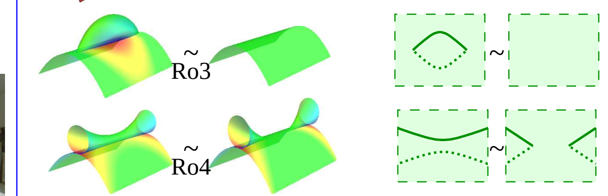
(only double curves are allowed in ribbon 2-knots)


"a triple point"

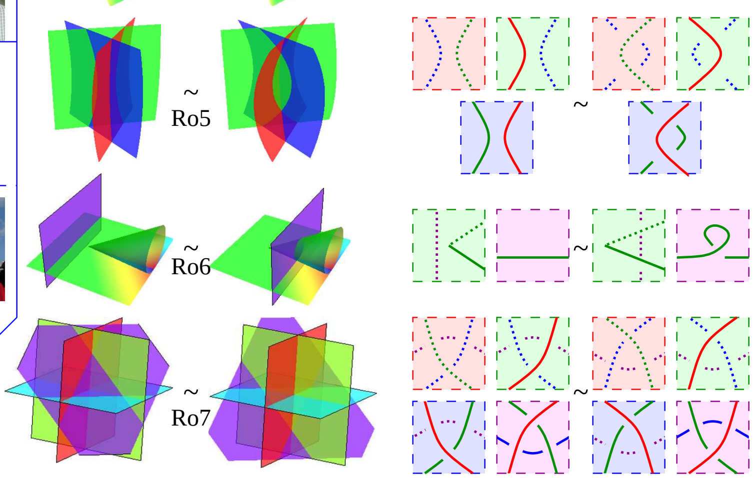

"a branch point"


Carter
Banach
Saito

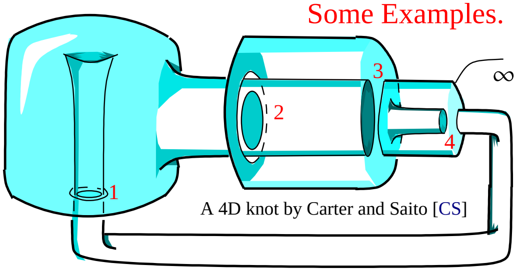




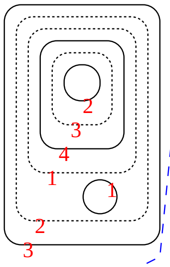

Roseman [Ro]



Some Examples.

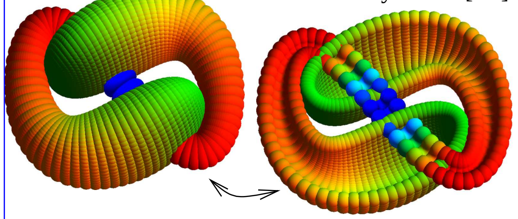


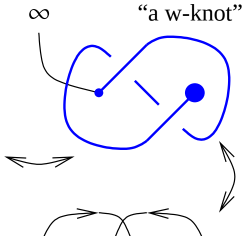
A 4D knot by Carter and Saito [CS]



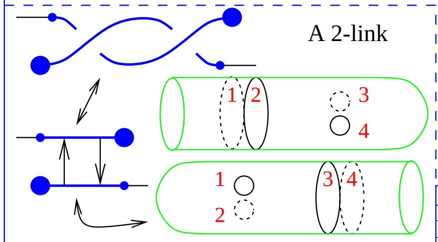

Dalvit

A 4D knot by Dalvit [Da]

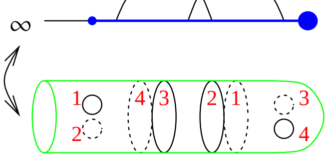




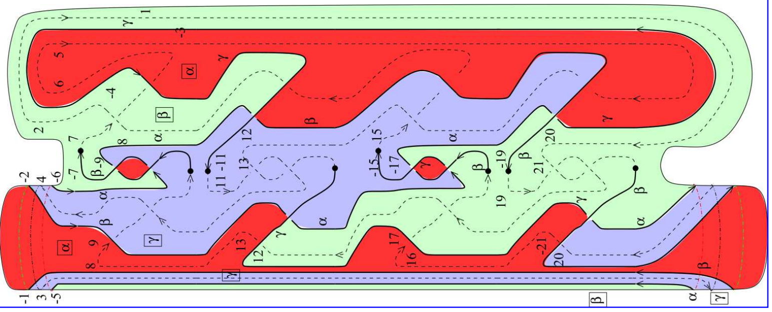
"a w-knot"



A 2-link



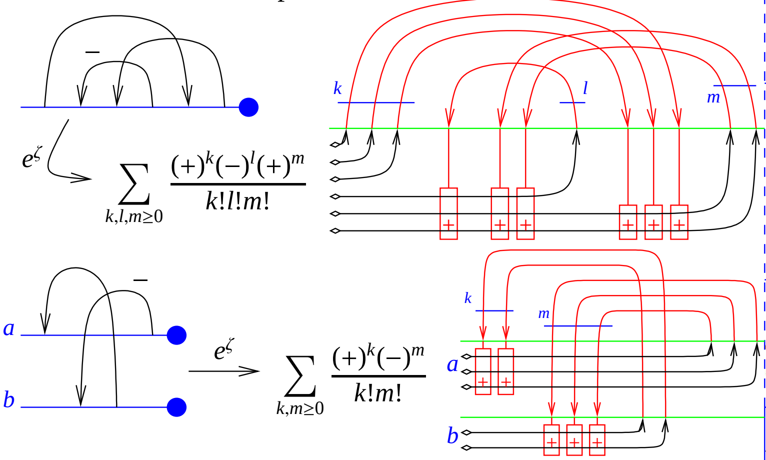
"ribbed cigar presentation"

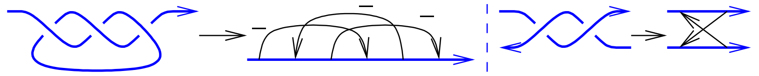


A 2-twist spun trefoil by Carter-Kamada-Saito [CKS].

A Partial Reduction of BF Theory to Combinatorics, 2

Theorem 1 (with Cattaneo, Dalvit (credit, no blame)). In the ribbon case, e^ζ can be computed as follows:





Theorem 2. Using Gauss diagrams to represent knots and T-component pure tangles, the above formulas define an invariant in $CW(FL(T)) \to CW(T)$, "cyclic words in T".
• Agrees with BN-Dancso [BND] and with [BN2]. • In-practice computable! • Vanishes on braids. • Extends to w. • Contains Alexander. • The "missing factor" in Levine's factorization [Le] (the rest of [Le] also fits, hence contains the MVA). • Related to / extends Farber's [Fa]? • Should be summed and categorified.

References.

[Ar] V. I. Arnold, *Topological Invariants of Plane Curves and Caustics*, University Lecture Series **5**, American Mathematical Society 1994.

[BN1] D. Bar-Natan, *Bracelets and the Goussarov filtration of the space of knots, Invariants of knots and 3-manifolds (Kyoto 2001)*, Geometry and Topology Monographs **4** 1–12, arXiv:math.GT/0111267.

[BN2] D. Bar-Natan, *Balloons and Hoops and their Universal Finite Type Invariant, BF Theory, and an Ultimate Alexander Invariant*, http://www.math.toronto.edu/~drorbn/papers/KBH/, arXiv:1308.1721.

[BND] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects: From Alexander to Kashiwara and Vergne*, http://www.math.toronto.edu/~drorbn/papers/WKO/.

[CKS] J. S. Carter, S. Kamada, and M. Saito, *Diagrammatic Computations for Quandles and Cocycle Knot Invariants*, Contemp. Math. **318** (2003) 51–74.

[CS] J. S. Carter and M. Saito, *Knotted surfaces and their diagrams*, Mathematical Surveys and Monographs **55**, American Mathematical Society, Providence 1998.

[Da] E. Dalvit, http://science.unitn.it/~dalvit/.

[CR] A. S. Cattaneo and C. A. Rossi, *Wilson Surfaces and Higher Dimensional Knot Invariants*, Commun. in Math. Phys. **256-3** (2005) 513–537, arXiv:math-ph/0210037.

[Fa] M. Farber, *Noncommutative Rational Functions and Boundary Links*, Math. Ann. **293** (1992) 543–568.

[Le] J. Levine, *A Factorization of the Conway Polynomial*, Comment. Math. Helv. **74** (1999) 27–53, arXiv:q-alg/9711007.

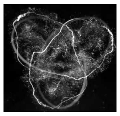
[Ro] D. Roseman, *Reidemeister-Type Moves for Surfaces in Four-Dimensional Space*, Knot Theory, Banach Center Publications **42** (1998) 347–380.

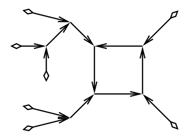
[Wa] T. Watanabe, *Configuration Space Integrals for Long n-Knots, the Alexander Polynomial and Knot Space Cohomology*, Alg. and Geom. Top. **7** (2007) 47-92, arXiv:math/0609742.

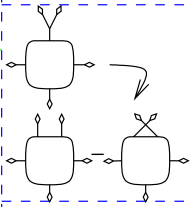
Continuing Joost Slingerland…

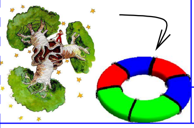
http://youtu.be/YCA0VIExVhge

http://youtu.be/mHyTOcfF99o

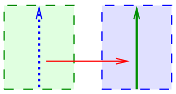


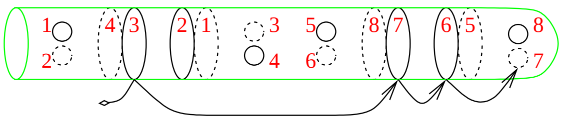






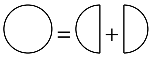
Sketch of Proof. In 4D axial gauge, only "drop down" red propagators, hence in the ribbon case, no M-trivalent vertices. S integrals are ±1 iff "ground pieces" run on nested curves as below, and exponentials arise when several propagators compete for the same double curve. And then the combinatorics is obvious…

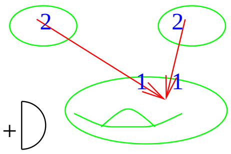


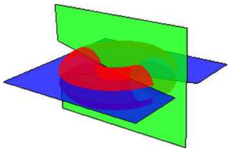


Musings

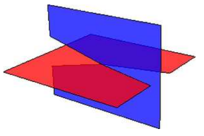
Chern-Simons. When the domain of BF is restricted to ribbon knots, and the target of Chern-Simons is restricted to trees and wheels, they agree. Why?

Is this all? What about the ∨-invariant? (the "true" triple linking number) 

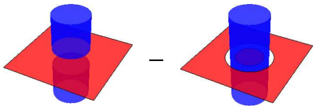




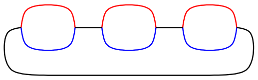
Gnots. In 3D, a generic immersion of S^1 is an embedding, a knot. In 4D, a generic immersion of a surface has finitely-many double points (a gnot?). Perhaps we should be studying these?



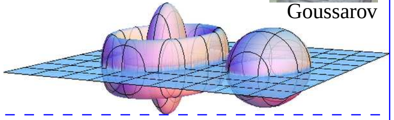
Finite type. What are finite-type invariants for 2-knots? What would be "chord diagrams"?



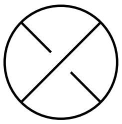
Bubble-wrap-finite-type. There's an alternative definition of finite type in 3D, due to Goussarov (see [BN1]). The obvious parallel in 4D involves "bubble wraps". Is it any good?

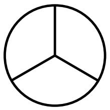


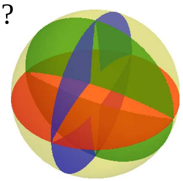

Goussarov

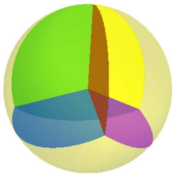


Shielded tangles. In 3D, one can't zoom in and compute "the Chern-Simons invariant of a tangle". Yet there are well-defined invariants of "shielded tangles", and rules for their compositions. What would the 4D analog be?



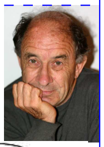

"an associator"

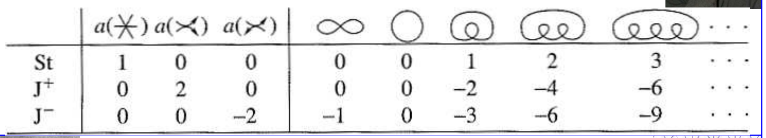




Will the relationship with the Kashiwara-Vergne problem [BND] necessarily arise here?

Plane curves. Shouldn't we understand integral / finite type invariants of plane curves, in the style of Arnold's J^+, J^-, and *St* [Ar], a bit better?


Arnold



	a(✳)	a(✕)	a(✕)	∞	○				· · ·
St	1	0	0	0	0	1	2	3	· · ·
J⁺	0	2	0	0	0	−2	−4	−6	· · ·
J⁻	0	0	−2	−1	0	−3	−6	−9	· · ·

"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

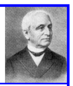
www.katlas.org

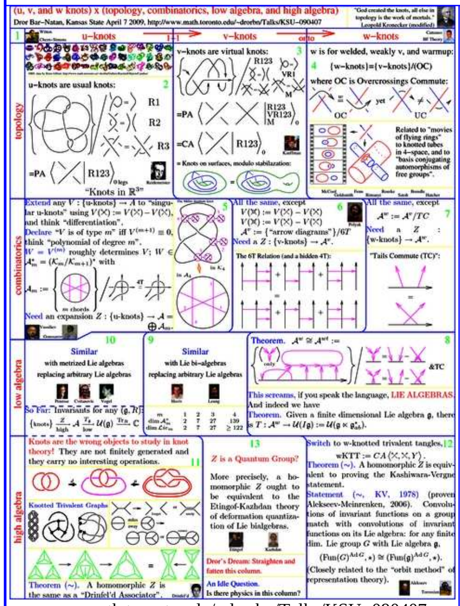
Convolutions on Lie Groups and Lie Algebras and Ribbon 2−Knots

Dror Bar−Natan, Bonn August 2009, http://www.math.toronto.edu/~drorbn/Talks/Bonn−0908

Disclaimer: Rough edges remain!

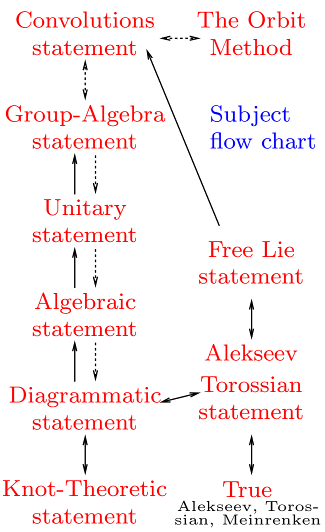
"God created the knots, all else in topology is the work of mortals." Leopold Kronecker (modified)





www.math.toronto.edu/~drorbn/Talks/KSU−090407

The Bigger Picture...



Convolutions statement

The Orbit Method

Group-Algebra statement

Subject flow chart

Unitary statement

Free Lie statement

Algebraic statement

Alekseev Torossian statement

Diagrammatic statement

Knot-Theoretic statement

True

Alekseev, Torossian, Meinrenken

Kashiwara

Vergne

Alekseev

Torossian

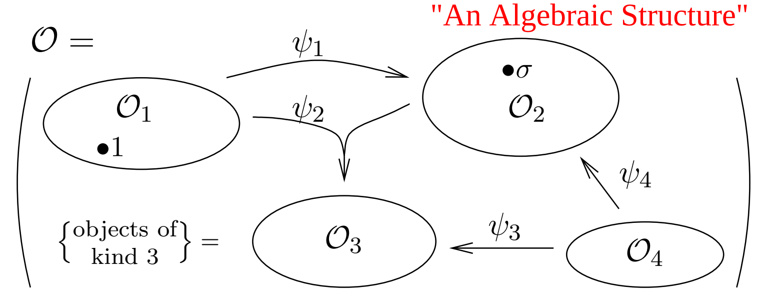
Homomorphic expansions for a filtered algebraic structure $\mathcal{K}$:

$$\begin{tikzcd}
\text{ops}\circlearrowright\mathcal{K} = \mathcal{K}_0 \arrow[d, Rightarrow] & \supset \mathcal{K}_1 & \supset \mathcal{K}_2 \arrow[d, "Z"] & \supset \mathcal{K}_3 & \supset \dots \\
\text{ops}\circlearrowright \operatorname{gr}\mathcal{K} := \mathcal{K}_0/\mathcal{K}_1 & \oplus\, \mathcal{K}_1/\mathcal{K}_2 & \oplus\, \mathcal{K}_2/\mathcal{K}_3 & \oplus\, \mathcal{K}_3/\mathcal{K}_4 & \oplus \dots
\end{tikzcd}$$

An **expansion** is a filtration respecting $Z : \mathcal{K} \to \operatorname{gr}\mathcal{K}$ that "covers" the identity on $\operatorname{gr}\mathcal{K}$. A **homomorphic expansion** is an expansion that respects all relevant "extra" operations.

Filtered algebraic structures are cheap and plenty. In any $\mathcal{K}$, allow formal linear combinations, let $\mathcal{K}_1$ be the ideal generated by differences (the "augmentation ideal"), and let $\mathcal{K}_m := \langle(\mathcal{K}_1)^m\rangle$ (using all available "products").

"An Algebraic Structure"



- Has kinds, objects, operations, and maybe constants.
- Perhaps subject to some axioms.
- We always allow formal linear combinations.

Example: Pure Braids. PB_n is generated by x_{ij}, "strand i goes around strand j once", modulo "Reidemeister moves". $A_n := \operatorname{gr} PB_n$ is generated by $t_{ij} := x_{ij} - 1$, modulo the $4T$ relations $[t_{ij}, t_{ik} + t_{jk}] = 0$ (and some lesser ones too). Much happens in A_n, including the Drinfel'd theory of associators.

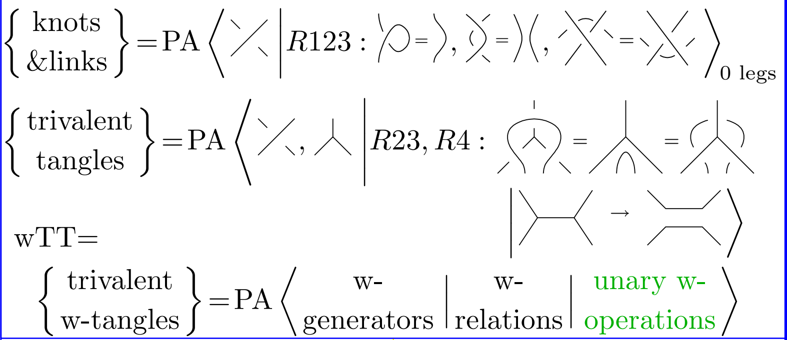
Our case(s).

$$\mathcal{K} \xrightarrow[\substack{\text{solving finitely many}\\ \text{equations in finitely}\\ \text{many unknowns}}]{Z:\ \text{high algebra}} \mathcal{A} := \operatorname{gr}\mathcal{K} \xrightarrow[\substack{\text{low algebra: pictures represent}\\ \text{formulas}}]{\text{given a "Lie" algebra } \mathfrak{g}} \text{"}\mathcal{U}(\mathfrak{g})\text{"}$$

$\mathcal{K}$ is knot theory or **topology**; $\operatorname{gr}\mathcal{K}$ is finite **combinatorics**: bounded-complexity diagrams modulo simple relations.

What are w-Trivalent Tangles?

(PA :=Planar Algebra)



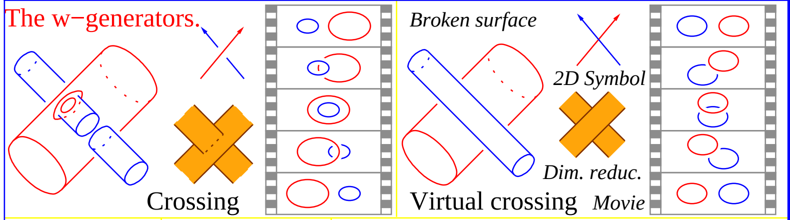
$\left\{\begin{matrix}\text{knots}\\ \text{\&links}\end{matrix}\right\} = \text{PA}\langle \ \mid R123 : \ \rangle_{\text{0 legs}}$

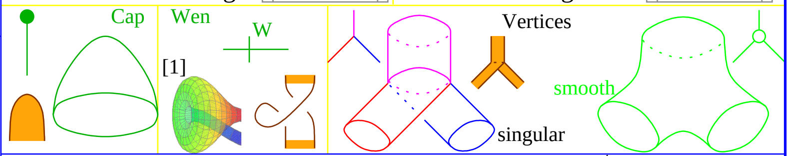
$\left\{\begin{matrix}\text{trivalent}\\ \text{tangles}\end{matrix}\right\} = \text{PA}\langle \ , \ \mid R23, R4 : \ \rangle$

wTT=

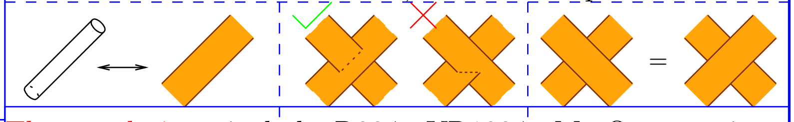
$\left\{\begin{matrix}\text{trivalent}\\ \text{w-tangles}\end{matrix}\right\} = \text{PA}\left\langle \text{w-generators} \mid \text{w-relations} \mid \text{unary w-operations} \right\rangle$

The w−generators.

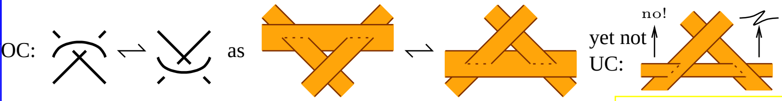


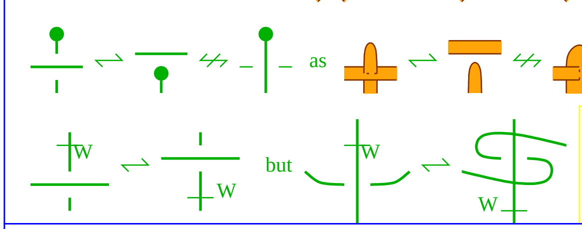


A Ribbon 2-Knot is a surface S embedded in $\mathbb{R}^4$ that bounds an immersed handlebody B, with only "ribbon singularities"; a ribbon singularity is a disk D of trasverse double points, whose preimages in B are a disk D_1 in the interior of B and a disk D_2 with $D_2 \cap \partial B = \partial D_2$, modulo isotopies of S alone.



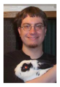
The w-relations include R234, VR1234, M, Overcrossings Commute (OC) but not UC, $W^2 = 1$, and funny interactions between the wen and the cap and over- and under-crossings:



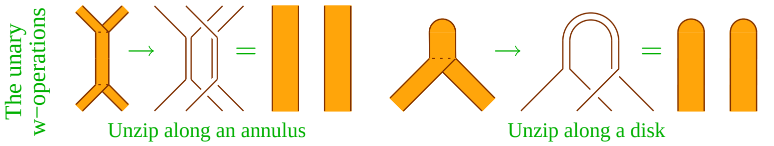


Challenge. Do the Reidemeister!

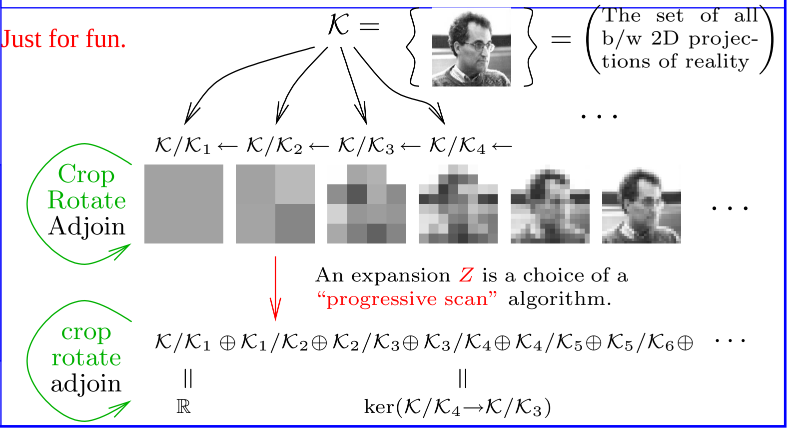
Reidemeister



Winter



Just for fun.



$\mathcal{K} = \{\ \} = \left(\begin{matrix}\text{The set of all}\\ \text{b/w 2D projec-}\\ \text{tions of reality}\end{matrix}\right)$

$\mathcal{K}/\mathcal{K}_1 \leftarrow \mathcal{K}/\mathcal{K}_2 \leftarrow \mathcal{K}/\mathcal{K}_3 \leftarrow \mathcal{K}/\mathcal{K}_4 \leftarrow \dots$

Crop Rotate Adjoin

An expansion Z is a choice of a "progressive scan" algorithm.

crop rotate adjoin

$\mathcal{K}/\mathcal{K}_1 \oplus \mathcal{K}_1/\mathcal{K}_2 \oplus \mathcal{K}_2/\mathcal{K}_3 \oplus \mathcal{K}_3/\mathcal{K}_4 \oplus \mathcal{K}_4/\mathcal{K}_5 \oplus \mathcal{K}_5/\mathcal{K}_6 \oplus \dots$

$\mathcal{K}/\mathcal{K}_1 = \mathbb{R}$, $\quad \mathcal{K}_3/\mathcal{K}_4 = \ker(\mathcal{K}/\mathcal{K}_4 \to \mathcal{K}/\mathcal{K}_3)$

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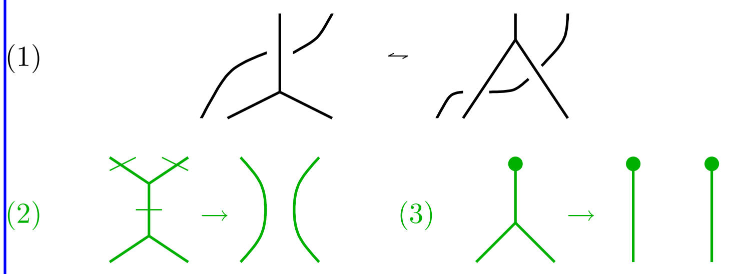
[1] http://qlink.queensu.ca/~4lb11/interesting.html

29/5/10, 8:42am

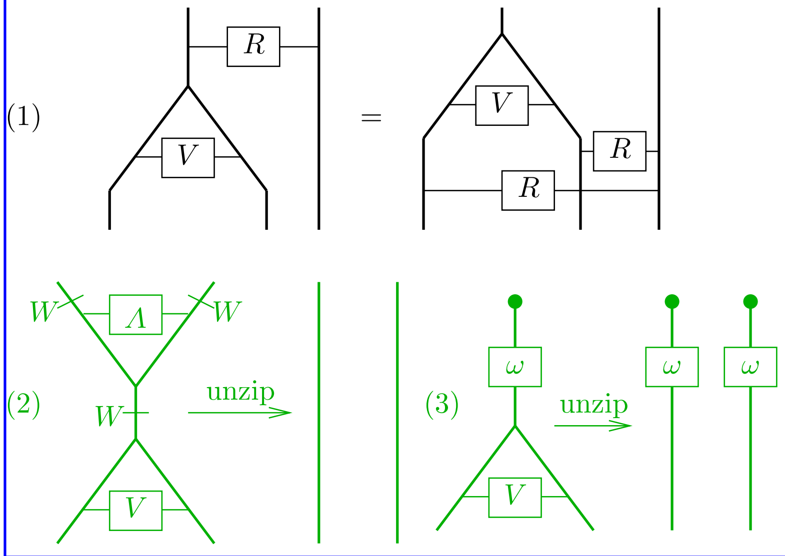
Also see http://www.math.toronto.edu/~drorbn/papers/WKO/

Convolutions on Lie Groups and Lie Algebras and Ribbon 2−Knots, Page 2

Knot-Theoretic statement. There exists a homomorphic expansion Z for trivalent w-tangles. In particular, Z should respect $R4$ and intertwine annulus and disk unzips:



Diagrammatic statement. Let $R = \exp \mapsto\!\!\!\!\uparrow \in \mathcal{A}^w(\uparrow\uparrow)$. There exist $\omega \in \mathcal{A}^w(\uparrow)$ and $V \in \mathcal{A}^w(\uparrow\uparrow)$ so that



Algebraic statement. With $I\mathfrak{g} := \mathfrak{g}^* \rtimes \mathfrak{g}$, with $c : \hat{\mathcal{U}}(I\mathfrak{g}) \to \hat{\mathcal{U}}(I\mathfrak{g})/\hat{\mathcal{U}}(\mathfrak{g}) = \hat{\mathcal{S}}(\mathfrak{g}^*)$ the obvious projection, with S the antipode of $\hat{\mathcal{U}}(I\mathfrak{g})$, with W the automorphism of $\hat{\mathcal{U}}(I\mathfrak{g})$ induced by flipping the sign of $\mathfrak{g}^*$, with $r \in \mathfrak{g}^* \otimes \mathfrak{g}$ the identity element and with $R = e^r \in \hat{\mathcal{U}}(I\mathfrak{g}) \otimes \hat{\mathcal{U}}(\mathfrak{g})$ there exist $\omega \in \hat{\mathcal{S}}(\mathfrak{g}^*)$ and $V \in \hat{\mathcal{U}}(I\mathfrak{g})^{\otimes 2}$ so that

(1) $V(\Delta \otimes 1)(R) = R^{13}R^{23}V$ in $\hat{\mathcal{U}}(I\mathfrak{g})^{\otimes 2} \otimes \hat{\mathcal{U}}(\mathfrak{g})$

(2) $V \cdot SWV = 1$ (3) $(c \otimes c)(V\Delta(\omega)) = \omega \otimes \omega$

Unitary statement. There exists $\omega \in \mathrm{Fun}(\mathfrak{g})^G$ and an (infinite order) tangential differential operator V defined on $\mathrm{Fun}(\mathfrak{g}_x \times \mathfrak{g}_y)$ so that

(1) $V\widehat{e^{x+y}} = \widehat{e^x}\widehat{e^y}V$ (allowing $\hat{\mathcal{U}}(\mathfrak{g})$-valued functions)

(2) $VV^* = I$ (3) $V\omega_{x+y} = \omega_x\omega_y$

Group-Algebra statement. There exists $\omega^2 \in \mathrm{Fun}(\mathfrak{g})^G$ so that for every $\phi, \psi \in \mathrm{Fun}(\mathfrak{g})^G$ (with small support), the following holds in $\hat{\mathcal{U}}(\mathfrak{g})$: (shhh, $\omega^2 = j^{1/2}$)

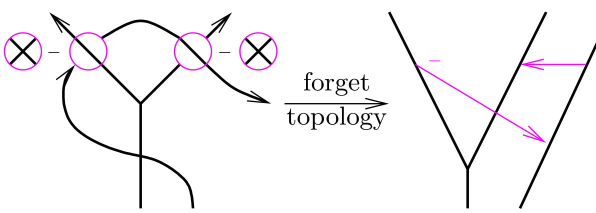
$$\iint_{\mathfrak{g}\times\mathfrak{g}} \phi(x)\psi(y)\omega^2_{x+y}e^{x+y} = \iint_{\mathfrak{g}\times\mathfrak{g}} \phi(x)\psi(y)\omega^2_x\omega^2_y e^x e^y.$$

(shhh, this is Duflo)

Convolutions statement (Kashiwara-Vergne). Convolutions of invariant functions on a Lie group agree with convolutions of invariant functions on its Lie algebra. More accurately, let G be a finite dimensional Lie group and let $\mathfrak{g}$ be its Lie algebra, let $j : \mathfrak{g} \to \mathbb{R}$ be the Jacobian of the exponential map $\exp : \mathfrak{g} \to G$, and let $\Phi : \mathrm{Fun}(G) \to \mathrm{Fun}(\mathfrak{g})$ be given by $\Phi(f)(x) := j^{1/2}(x)f(\exp x)$. Then if $f, g \in \mathrm{Fun}(G)$ are Ad-invariant and supported near the identity, then

$$\Phi(f) \star \Phi(g) = \Phi(f \star g).$$

From wTT to $\mathcal{A}^w$. $\mathrm{gr}_m\,\mathrm{wTT} := \{m\text{−cubes}\}/\{(m+1)\text{−cubes}\}$:

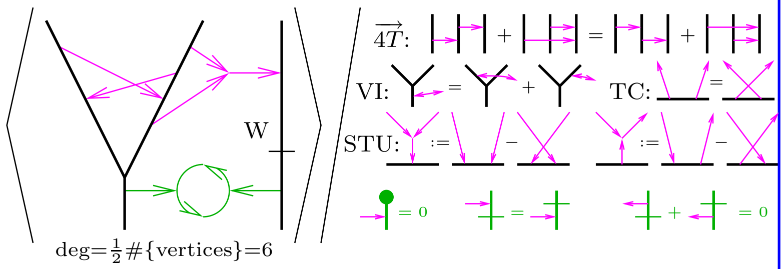


Vassiliev

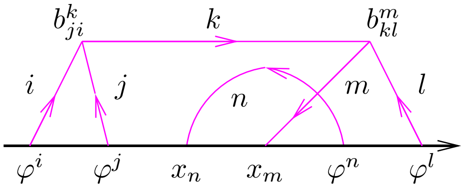
Polyak

Goussarov

w-Jacobi diagrams and $\mathcal{A}$. $\mathcal{A}^w(Y\uparrow) \cong \mathcal{A}^w(\uparrow\uparrow\uparrow)$ is



Diagrammatic to Algebraic. With (x_i) and (φ^j) dual bases of $\mathfrak{g}$ and $\mathfrak{g}^*$ and with $[x_i, x_j] = \sum b^k_{ij}x_k$, we have $\mathcal{A}^w \to \mathcal{U}$ via



Penrose

Cvitanovic

$$\longrightarrow \sum_{i,j,k,l,m,n=1}^{\dim \mathfrak{g}} b^k_{ij}b^m_{kl}\varphi^i\varphi^j x_n x_m \varphi^l \in \mathcal{U}(I\mathfrak{g})$$

Unitary $\Longleftrightarrow$ Algebraic. The key is to interpret $\hat{\mathcal{U}}(I\mathfrak{g})$ as tangential differential operators on $\mathrm{Fun}(\mathfrak{g})$:

- $\varphi \in \mathfrak{g}^*$ becomes a multiplication operator.
- $x \in \mathfrak{g}$ becomes a tangential derivation, in the direction of the action of $\mathrm{ad}\,x$: $(x\varphi)(y) := \varphi([x, y])$.
- $c : \hat{\mathcal{U}}(I\mathfrak{g}) \to \hat{\mathcal{U}}(I\mathfrak{g})/\hat{\mathcal{U}}(\mathfrak{g}) = \hat{\mathcal{S}}(\mathfrak{g}^*)$ is “the constant term”.

Unitary $\Longrightarrow$ Group-Algebra. $\iint \omega^2_{x+y}e^{x+y}\phi(x)\psi(y)$

$$= \langle \omega_{x+y}, \omega_{x+y}e^{x+y}\phi(x)\psi(y)\rangle = \langle V\omega_{x+y}, Ve^{x+y}\phi(x)\psi(y)\omega_{x+y}\rangle$$
$$= \langle \omega_x\omega_y, e^xe^yV\phi(x)\psi(y)\omega_{x+y}\rangle = \langle \omega_x\omega_y, e^xe^y\phi(x)\psi(y)\omega_x\omega_y\rangle$$
$$= \iint \omega^2_x\omega^2_y e^xe^y\phi(x)\psi(y).$$

Convolutions and Group Algebras (ignoring all Jacobians). If G is finite, A is an algebra, $\tau : G \to A$ is multiplicative then $(\mathrm{Fun}(G), \star) \cong (A, \cdot)$ via $L : f \mapsto \sum f(a)\tau(a)$. For Lie $(G, \mathfrak{g})$,

$$\begin{tikzcd}
(\mathfrak{g},+) \ni x \arrow[r, "\tau_0=\exp_{\mathcal{S}}"] \arrow[d, "\exp_G"] \arrow[dr, "\exp_{\mathcal{U}}"] & e^x \in \hat{\mathcal{S}}(\mathfrak{g}) \arrow[d, "\chi"] & \text{so} & \mathrm{Fun}(\mathfrak{g}) \arrow[r, "L_0"] \arrow[d, "\Phi^{-1}"] & \hat{\mathcal{S}}(\mathfrak{g}) \arrow[d, "\chi"] \\
(G,\cdot) \ni e^x \arrow[r, "\tau_1"] & e^x \in \hat{\mathcal{U}}(\mathfrak{g}) & & \mathrm{Fun}(G) \arrow[r, "L_1"] & \hat{\mathcal{U}}(\mathfrak{g})
\end{tikzcd}$$

with $L_0\psi = \int \psi(x)e^x dx \in \hat{\mathcal{S}}(\mathfrak{g})$ and $L_1\Phi^{-1}\psi = \int \psi(x)e^x \in \hat{\mathcal{U}}(\mathfrak{g})$. Given $\psi_i \in \mathrm{Fun}(\mathfrak{g})$ compare $\Phi^{-1}(\psi_1) \star \Phi^{-1}(\psi_2)$ and $\Phi^{-1}(\psi_1 \star \psi_2)$ in $\hat{\mathcal{U}}(\mathfrak{g})$: (shhh, $L_{0/1}$ are “Laplace transforms”)

$$\star \text{ in } G : \iint \psi_1(x)\psi_2(y)e^xe^y \qquad \star \text{ in } \mathfrak{g} : \iint \psi_1(x)\psi_2(y)e^{x+y}$$

We skipped...

- The Alexander polynomial and Milnor numbers.
- u-Knots, Alekseev-Torossian, and Drinfel’d associators.
- v-Knots, quantum groups and Etingof-Kazhdan.
- BF theory and the successful religion of path integrals.
- The simplest problem hyperbolic geometry solves.