

Pensieve header: Mathematica notebook accompanying "A Note on the Unitarity Property of the Gassner Invariant" by Dror Bar-Natan, <http://drorbn.net/AcademicPensieve/2014-06/UnitarityOf-Gassner/>.

Definitions.

```

U_i[t_] := ReplacePart[IdentityMatrix[n],
  {{i, i} -> 1 - t, {i, i + 1} -> 1,
   {i + 1, i} -> t, {i + 1, i + 1} -> 0}];
U_i_bar[t_] := Inverse[U_i[t]];
Omega[tau___] := Table[
  Which[i < j, 0, i == j, (1 - t_{tau}[[i]])^-1, i > j, 1],
  {i, n}, {j, n}];
X_bar := X /. t_i -> 1 / t_i;
U_i,j_ := ReplacePart[IdentityMatrix[n],
  {{i, i} -> 1, {i, j} -> 1 - t_i,
   {j, i} -> 0, {j, j} -> t_i}];
V_i,j_ := ReplacePart[IdentityMatrix[n],
  {{i, i} -> 1, {i, j} -> 1 - t_j,
   {j, i} -> 0, {j, j} -> t_i}];
DD := DiagonalMatrix[Table[1 - t_i, {i, n}]];

```

The named matrices.

```
n = 5; MatrixForm /@ Simplify /@ {U_3[t], U_3_bar[t]}
```

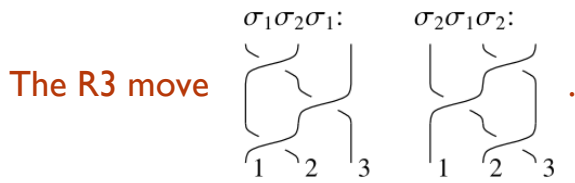
$$\left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1-t & 1 & 0 \\ 0 & 0 & t & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{t} & 0 \\ 0 & 0 & 1 & \frac{-1+t}{t} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right\}$$

```
n = 3; MatrixForm /@ Simplify /@ {Omega[2, 3, 1], Inverse[Omega[2, 3, 1]]}
```

$$\left\{ \begin{pmatrix} \frac{1}{1-t_2} & 0 & 0 \\ 1 & \frac{1}{1-t_3} & 0 \\ 1 & 1 & \frac{1}{1-t_1} \end{pmatrix}, \begin{pmatrix} 1-t_2 & 0 & 0 \\ -(-1+t_2)(-1+t_3) & 1-t_3 & 0 \\ -(-1+t_1)(-1+t_2)t_3 & -(-1+t_1)(-1+t_3) & 1-t_1 \end{pmatrix} \right\}$$

```
n = 5; MatrixForm /@ {U_4,1, V_4,1, DD}
```

$$\left\{ \begin{pmatrix} t_4 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1-t_4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} t_4 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1-t_1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1-t_1 & 0 & 0 & 0 & 0 \\ 0 & 1-t_2 & 0 & 0 & 0 \\ 0 & 0 & 1-t_3 & 0 & 0 \\ 0 & 0 & 0 & 1-t_4 & 0 \\ 0 & 0 & 0 & 0 & 1-t_5 \end{pmatrix} \right\}$$



```
n = 3; MatrixForm /@ Simplify /@ {U1[t1].U2[t1].U1[t2], U2[t2].U1[t1].U2[t1]}
```

$$\left\{ \begin{pmatrix} 1-t_1 & 1-t_1 & 1 \\ -t_1(-1+t_2) & t_1 & 0 \\ t_1 t_2 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1-t_1 & 1-t_1 & 1 \\ -t_1(-1+t_2) & t_1 & 0 \\ t_1 t_2 & 0 & 0 \end{pmatrix} \right\}$$

The unitarity property for the generators.

```
n = 5; γ = U3[t3];
```

```
MatrixForm /@
```

```
Simplify /@ {Ω[1, 2, 4, 3, 5].Inverse[γ], Transpose[γ̄].Ω[1, 2, 3, 4, 5]}
```

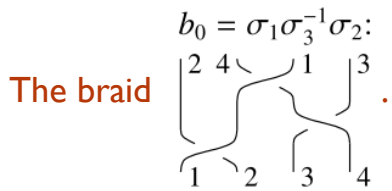
$$\left\{ \begin{pmatrix} \frac{1}{1-t_1} & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{1-t_2} & 0 & 0 & 0 \\ 1 & 1 & 0 & \frac{1}{t_3-t_3 t_4} & 0 \\ 1 & 1 & \frac{1}{1-t_3} & 0 & 0 \\ 1 & 1 & 1 & 1 & \frac{1}{1-t_5} \end{pmatrix}, \begin{pmatrix} \frac{1}{1-t_1} & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{1-t_2} & 0 & 0 & 0 \\ 1 & 1 & 0 & \frac{1}{t_3-t_3 t_4} & 0 \\ 1 & 1 & \frac{1}{1-t_3} & 0 & 0 \\ 1 & 1 & 1 & 1 & \frac{1}{1-t_5} \end{pmatrix} \right\}$$

```
n = 5; γ = U3̄[t4];
```

```
MatrixForm /@
```

```
FullSimplify /@ {Ω[1, 2, 4, 3, 5].Inverse[γ], Transpose[γ̄].Ω[1, 2, 3, 4, 5]}
```

$$\left\{ \begin{pmatrix} \frac{1}{1-t_1} & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{1-t_2} & 0 & 0 & 0 \\ 1 & 1 & 1 & \frac{1}{1-t_4} & 0 \\ 1 & 1 & 1 - \frac{t_3 t_4}{-1+t_3} & 1 & 0 \\ 1 & 1 & 1 & 1 & \frac{1}{1-t_5} \end{pmatrix}, \begin{pmatrix} \frac{1}{1-t_1} & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{1-t_2} & 0 & 0 & 0 \\ 1 & 1 & 1 & \frac{1}{1-t_4} & 0 \\ 1 & 1 & 1 - \frac{t_3 t_4}{-1+t_3} & 1 & 0 \\ 1 & 1 & 1 & 1 & \frac{1}{1-t_5} \end{pmatrix} \right\}$$



```
n = 4; MatrixForm[γ₀ = U₁[t₁].U₃[t₄].U₂[t₁]]
```

$$\begin{pmatrix} 1-t_1 & 1-t_1 & 1 & 0 \\ t_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{t_4} \\ 0 & t_1 & 0 & -\frac{1-t_4}{t_4} \end{pmatrix}$$

The unitarity property for b_0 .

```
MatrixForm /@ Simplify /@ {Ω[2, 4, 1, 3].Inverse[γ₀], Transpose[γ̄₀].Ω[1, 2, 3, 4]}
```

$$\left\{ \begin{pmatrix} 0 & \frac{1}{t_1-t_1 t_2} & 0 & 0 \\ 0 & \frac{1}{t_1} & \frac{1}{t_1} & \frac{1}{t_1-t_1 t_4} \\ \frac{1}{1-t_1} & 0 & 0 & 0 \\ 1 & 1 & -\frac{1+t_3(-1+t_4)}{-1+t_3} & 1 \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{t_1-t_1 t_2} & 0 & 0 \\ 0 & \frac{1}{t_1} & \frac{1}{t_1} & \frac{1}{t_1-t_1 t_4} \\ \frac{1}{1-t_1} & 0 & 0 & 0 \\ 1 & 1 & -\frac{1+t_3(-1+t_4)}{-1+t_3} & 1 \end{pmatrix} \right\}$$

On to w-braids

```
n = 3; MatrixForm /@ Simplify /@ {U₁,₂.U₁,₃.U₂,₃, U₂,₃.U₁,₃.U₁,₂}
```

$$\left\{ \begin{pmatrix} 1 & 1-t_1 & 1-t_1 \\ 0 & t_1 & -t_1(-1+t_2) \\ 0 & 0 & t_1 t_2 \end{pmatrix}, \begin{pmatrix} 1 & 1-t_1 & 1-t_1 \\ 0 & t_1 & -t_1(-1+t_2) \\ 0 & 0 & t_1 t_2 \end{pmatrix} \right\}$$

```
n = 3; MatrixForm /@ Simplify /@ {U₁,₂.U₁,₃, U₁,₃.U₁,₂}
```

$$\left\{ \begin{pmatrix} 1 & 1-t_1 & 1-t_1 \\ 0 & t_1 & 0 \\ 0 & 0 & t_1 \end{pmatrix}, \begin{pmatrix} 1 & 1-t_1 & 1-t_1 \\ 0 & t_1 & 0 \\ 0 & 0 & t_1 \end{pmatrix} \right\}$$

The “Other Gassner” Γ

```
n = 4; MatrixForm /@ Simplify /@ {V₄,₁, Inverse[DD].U₄,₁.DD}
```

$$\left\{ \begin{pmatrix} t_4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1-t_1 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} t_4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1-t_1 & 0 & 0 & 1 \end{pmatrix} \right\}$$

```
n = 4; MatrixForm /@ Simplify /@ {
  Transpose[DD].Ω[1, 2, 3, 4].DD,
  Ψ' = i Transpose[DD].(Ω[1, 2, 3, 4] - Transpose[Ω[1, 2, 3, 4]]).DD
}
```

$$\left(\begin{array}{cccc} 1 - \frac{1}{t_1} & 0 & 0 & 0 \\ (1 - t_1) \left(1 - \frac{1}{t_2}\right) & 1 - \frac{1}{t_2} & 0 & 0 \\ (1 - t_1) \left(1 - \frac{1}{t_3}\right) & (1 - t_2) \left(1 - \frac{1}{t_3}\right) & 1 - \frac{1}{t_3} & 0 \\ (1 - t_1) \left(1 - \frac{1}{t_4}\right) & (1 - t_2) \left(1 - \frac{1}{t_4}\right) & (1 - t_3) \left(1 - \frac{1}{t_4}\right) & 1 - \frac{1}{t_4} \end{array} \right),$$

$$\left(\begin{array}{cccc} \frac{i(-1+t_1^2)}{t_1} & i\left(-1 + \frac{1}{t_1}\right)(1-t_2) & i\left(-1 + \frac{1}{t_1}\right)(1-t_3) & i\left(-1 + \frac{1}{t_1}\right)(1-t_4) \\ -\frac{i(-1+t_1)(-1+t_2)}{t_2} & \frac{i(-1+t_2^2)}{t_2} & i\left(-1 + \frac{1}{t_2}\right)(1-t_3) & i\left(-1 + \frac{1}{t_2}\right)(1-t_4) \\ -\frac{i(-1+t_1)(-1+t_3)}{t_3} & -\frac{i(-1+t_2)(-1+t_3)}{t_3} & \frac{i(-1+t_3^2)}{t_3} & i\left(-1 + \frac{1}{t_3}\right)(1-t_4) \\ -\frac{i(-1+t_1)(-1+t_4)}{t_4} & -\frac{i(-1+t_2)(-1+t_4)}{t_4} & -\frac{i(-1+t_3)(-1+t_4)}{t_4} & \frac{i(-1+t_4^2)}{t_4} \end{array} \right)$$