Pensieve header: Testing the common program for all w-meta-calculi. Continues pensieve://2014-05/MetaCalculi/, continued pensieve://2014-07/MetaCalculi/.

```
dir = SetDirectory["C:/drorbn/AcademicPensieve/2014-06/MetaCalculi/"];
<< KnotTheory`
<< MetaCalculi-Program.m
```

```
Loading KnotTheory` version of April 3, 2014, 16:23:56.0784.
Read more at http://katlas.org/wiki/KnotTheory.
```

General

```
SXForm[L = Link["L6a4"]]
```

KnotTheory::loading : Loading precomputed data in PD4Links`.

```
SXForm[{Loop[1, 2, 3, 4], Loop[5, 6, 7, 8], Loop[9, 10, 11, 12]},
 Xm[1, 6] Xm[5, 10] Xm[9, 2] Xp[3, 8] Xp[7, 12] Xp[11, 4]]
```

```
Z[L]
```

```
dm[9, 12, 9][
 dm[9, 11, 9][dm[9, 10, 9][dm[5, 8, 5][dm[5, 7, 5][dm[5, 6, 5][dm[1, 4, 1][dm[1, 3, 1][
        dm[1, 2, 1][Xm[1, 6] Xm[5, 10] Xm[9, 2] Xp[3, 8] Xp[7, 12] Xp[11, 4]]]]]]]]]]
```

α-Calculus

```
{Xp_ab, Xm_ab} // A
```

$$\left\{\begin{pmatrix} 1 & h[b] \\ t[a] & \frac{-1+e^{c_a}}{c_a} \end{pmatrix}, \begin{pmatrix} 1 & h[b] \\ t[a] & \frac{e^{-c_a}(1-e^{c_a})}{c_a} \end{pmatrix}\right\}$$

```
{Xm_51 Xm_62 Xp_34 // A // dm[1, 4, 1] // dm[2, 5, 2] // dm[3, 6, 3],
 Xp_61 Xm_24 Xm_35 // A // dm[1, 4, 1] // dm[2, 5, 2] // dm[3, 6, 3]}
```

$$\left\{\begin{pmatrix} 1 & h[1] & h[2] \\ t[2] & \frac{e^{-c_2}(1-e^{c_2})}{c_2} & 0 \\ t[3] & \frac{e^{-c_2}(-1+e^{c_3})}{c_3} & \frac{e^{-c_3}(1-e^{c_3})}{c_3} \end{pmatrix}, \begin{pmatrix} 1 & h[1] & h[2] \\ t[2] & \frac{e^{-c_2}(1-e^{c_2})}{c_2} & 0 \\ t[3] & \frac{e^{-c_2}(-1+e^{c_3})}{c_3} & \frac{e^{-c_3}(1-e^{c_3})}{c_3} \end{pmatrix}\right\}$$

```
α = Xm12,1 Xm27 Xm83 Xm4,11 Xp16,5 Xp6,13 Xp14,9 Xp10,15 // A
```

1	h[1]	h[3]	h[5]	h[7]	h[9]	h[11]	h[13]	h[15]
t[2]	0	0	0	$\frac{e^{-c_2}(1-e^{c_2})}{c_2}$	0	0	0	0
t[4]	0	0	0	0	0	$\frac{e^{-c_4}(1-e^{c_4})}{c_4}$	0	0
t[6]	0	0	0	0	0	0	$\frac{-1+e^{c_6}}{c_6}$	0
t[8]	0	$\frac{e^{-c_8}(1-e^{c_8})}{c_8}$	0	0	0	0	0	0
t[10]	0	0	0	0	0	0	0	$\frac{-1+e^{c_{10}}}{c_{10}}$
t[12]	$\frac{e^{-c_{12}}(1-e^{c_{12}})}{c_{12}}$	0	0	0	0	0	0	0
t[14]	0	0	0	0	$\frac{-1+e^{c_{14}}}{c_{14}}$	0	0	0
t[16]	0	0	$\frac{-1+e^{c_{16}}}{c_{16}}$	0	0	0	0	0

```
α = Xm12,1 Xm27 Xm83 Xm4,11 Xp16,5 Xp6,13 Xp14,9 Xp10,15 // A;
Do[α = α // dm[1, k, 1], {k, 2, 16}]; α
```

$$\begin{pmatrix} e^{-3c_1}\left(-1+4e^{c_1}-8e^{2c_1}+11e^{3c_1}-8e^{4c_1}+4e^{5c_1}-e^{6c_1}\right) \\ t[1] \end{pmatrix}$$

Testing R3

```
{(Xp12 // A) ** (Xp13 // A) ** (Xp23 // A), (Xp23 // A) ** (Xp13 // A) ** (Xp12 // A)}
```

$$\left\{\begin{pmatrix} 1 & h[2] & h[3] \\ t[1] & \frac{-1+e^{c_1}}{c_1} & \frac{-1+e^{c_1}}{c_1} \\ t[2] & 0 & \frac{-e^{c_1}+e^{c_1+c_2}}{c_2} \end{pmatrix}, \begin{pmatrix} 1 & h[2] & h[3] \\ t[1] & \frac{-1+e^{c_1}}{c_1} & \frac{-1+e^{c_1}}{c_1} \\ t[2] & 0 & \frac{-e^{c_1}+e^{c_1+c_2}}{c_2} \end{pmatrix}\right\}$$

Testing the KV Solution

The Hard R4 Equation

```
Print /@ {(Xp23 // A) ** (Xp13 // A) ** (V // A), (V // A) ** ((Xp13 // A) // dΔ[1, 1, 2])};
```

$$\left(\begin{array}{ccc} \frac{2^{1/4}\left(\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}\right)^{1/4}\left(\frac{\text{Sinh}\left[\frac{c_2}{2}\right]}{c_2}\right)^{1/4}}{\left(\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}\right)^{1/4}} & \text{h}[1] & \\ \text{t}[1] & \frac{-\sqrt{2}\sqrt{\frac{\text{Sinh}\left[\frac{c_2}{2}\right]}{c_2}}\,c_2+2\sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_2\sqrt{\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}}{2\sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_1^2\sqrt{\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}+2\sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_1\,c_2\sqrt{\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}} & \\ \text{t}[2] & \frac{\sqrt{2}\sqrt{\frac{\text{Sinh}\left[\frac{c_2}{2}\right]}{c_2}}-2\sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\sqrt{\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}}{2\sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_1\sqrt{\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}+2\sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_2\sqrt{\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}} & \frac{-e^{\frac{c_1}{2}}\,c_1\sqrt{\frac{\text{Sinh}\left[\frac{c_2}{2}\right]}{c_2}}+e^{\frac{3c_1}{2}+c_2}\,c_1\sqrt{\frac{\text{Sinh}\left[\frac{c_2}{2}\right]}{c_2}}}{} \end{array}\right)$$

$$\left(\begin{array}{ccc} \frac{2^{1/4}\left(\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}\right)^{1/4}\left(\frac{\text{Sinh}\left[\frac{c_2}{2}\right]}{c_2}\right)^{1/4}}{\left(\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}\right)^{1/4}} & \text{h}[1] & \\ \text{t}[1] & \frac{-\sqrt{2}\sqrt{\frac{\text{Sinh}\left[\frac{c_2}{2}\right]}{c_2}}\,c_2+2\sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_2\sqrt{\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}}{2\sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_1^2\sqrt{\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}+2\sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_1\,c_2\sqrt{\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}} & \\ \text{t}[2] & \frac{\sqrt{2}\sqrt{\frac{\text{Sinh}\left[\frac{c_2}{2}\right]}{c_2}}-2\sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\sqrt{\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}}{2\sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_1\sqrt{\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}+2\sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_2\sqrt{\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}} & \frac{-e^{\frac{c_1}{2}}\,c_1\sqrt{\frac{\text{Sinh}\left[\frac{c_2}{2}\right]}{c_2}}+e^{\frac{3c_1}{2}+c_2}\,c_1\sqrt{\frac{\text{Sinh}\left[\frac{c_2}{2}\right]}{c_2}}}{} \end{array}\right)$$

```
(Xp23 // A) ** (Xp13 // A) ** (V // A) ==
  (V // A) ** ((Xp13 // A) // dΔ[1, 1, 2]) // Simplify
```

```
True
```

Unitarity

```
(V // A) ** (Vi // A)  // αCollect[Simplify]
```

$$(1)$$

qΔ (“renormalized cabling”)

```
qΔ0[z_, x_, y_][α_A] := Module[{b, a},
  Times[
      V // A // dσ[1 → b[x], 2 → b[y]],
      α // dΔ[z, x, y],
      V // A // dA[1] // dA[2] // dσ[1 → a[x], 2 → a[y]]
     ] // dm[b[x], x, x] // dm[b[y], y, y] // dm[x, a[x], x] // dm[y, a[y], y]
 ]
```

Xp_{13} // A // qΔ0[1, 1, 2] // αCollect[Simplify]

$$\begin{pmatrix} 1 & h[3] \\ t[1] & \frac{-e^{c_2}+e^{c_1+c_2}}{c_1} \\ t[2] & \frac{-1+e^{c_2}}{c_2} \end{pmatrix}$$

(Xp_{23} // A) ** (Xp_{13} // A)

$$\begin{pmatrix} 1 & h[3] \\ t[1] & \frac{-e^{c_2}+e^{c_1+c_2}}{c_1} \\ t[2] & \frac{-1+e^{c_2}}{c_2} \end{pmatrix}$$

Xp_{13} // A // qΔ[1, 1, 2] // αCollect[Simplify]

$$\begin{pmatrix} 1 & h[3] \\ t[1] & \frac{-e^{c_2}+e^{c_1+c_2}}{c_1+c_2} \\ t[2] & \frac{-1+e^{c_2}}{c_2} \end{pmatrix}$$

Clear[ω, α, θ, ϕ, Ξ];

$\alpha0 = A\left[\omega, \{t[a], t[S]\}.\begin{pmatrix} \alpha & \theta \\ \phi & \Xi \end{pmatrix}.\{h[a], h[S]\}\right];$

α1 = α0 // qΔ0[a, x, y] // αCollect[Simplify]

$$\begin{pmatrix} \omega & h[S] & h[x] & h[\\ t[S] & \Xi & \phi & \varphi \\ t[x] & \frac{-e^{c_y}\,\theta\,c_x+e^{c_x+c_y}\,\theta\,c_x-e^{c_y}\,\theta\,c_y+e^{c_x+c_y}\,\theta\,c_y}{-c_x+e^{c_x+c_y}\,c_x} & \frac{-e^{c_y}\,\alpha\,c_x+e^{c_x+c_y}\,\alpha\,c_x-e^{c_y}\,\alpha\,c_y+e^{c_x+c_y}\,\alpha\,c_y}{-c_x+e^{c_x+c_y}\,c_x} & \frac{-e^{c_y}\,\alpha\,c_x+e^{c_x+c_y}\,\alpha\,c_x}{-c_x+e^{} } \\ t[y] & \frac{-\theta\,c_x+e^{c_y}\,\theta\,c_x-\theta\,c_y+e^{c_y}\,\theta\,c_y}{-c_y+e^{c_x+c_y}\,c_y} & \frac{-\alpha\,c_x+e^{c_y}\,\alpha\,c_x-\alpha\,c_y+e^{c_y}\,\alpha\,c_y}{-c_y+e^{c_x+c_y}\,c_y} & \frac{-\alpha\,c_x+e^{c_y}\,\alpha\,c_x}{-c_y+e^{}} \end{pmatrix}$$

$\alpha Collect\left[\left(Simplify[\#] \;/.\; c_x + c_y \to c_a\right) \&\right][\alpha1]$

$$\begin{pmatrix} \omega & h[S] & h[x] & h[y] \\ t[a] & 0 & 0 & 0 \\ t[S] & \Xi & \phi & \phi \\ t[x] & \frac{-e^{c_y}\,\theta\,c_a+e^{c_x+c_y}\,\theta\,c_a}{-c_x+e^{c_a}\,c_x} & \frac{-e^{c_y}\,\alpha\,c_a+e^{c_x+c_y}\,\alpha\,c_a}{-c_x+e^{c_a}\,c_x} & \frac{-e^{c_y}\,\alpha\,c_a+e^{c_x+c_y}\,\alpha\,c_a}{-c_x+e^{c_a}\,c_x} \\ t[y] & \frac{-\theta\,c_a+e^{c_y}\,\theta\,c_a}{-c_y+e^{c_a}\,c_y} & \frac{-\alpha\,c_a+e^{c_y}\,\alpha\,c_a}{-c_y+e^{c_a}\,c_y} & \frac{-\alpha\,c_a+e^{c_y}\,\alpha\,c_a}{-c_y+e^{c_a}\,c_y} \end{pmatrix}$$

Clear[ω, α, θ, ϕ, Ξ];

$\alpha0 = A\left[\omega, \{t[a], t[S]\}.\begin{pmatrix} \alpha & \theta \\ \phi & \Xi \end{pmatrix}.\{h[a], h[S]\}\right];$

α0 // qΔ[a, x, y] // αCollect[Simplify]

$$\begin{pmatrix} \omega & h[S] & h[x] & h[y] \\ t[S] & \Xi & \phi & \phi \\ t[x] & \frac{-e^{c_y}\,\theta\,c_x+e^{c_x+c_y}\,\theta\,c_x-e^{c_y}\,\theta\,c_y+e^{c_x+c_y}\,\theta\,c_y}{-c_x+e^{c_x+c_y}\,c_x} & \frac{-e^{c_y}\,\alpha\,c_x+e^{c_x+c_y}\,\alpha\,c_x-e^{c_y}\,\alpha\,c_y+e^{c_x+c_y}\,\alpha\,c_y}{-c_x+e^{c_x+c_y}\,c_x} & \frac{-e^{c_y}\,\alpha\,c_x+e^{c_x+c_y}\,\alpha\,c_x-e^{c_y}\,\alpha\,c_y+e^{c_x+c_y}\,\alpha\,c_y}{-c_x+e^{c_x+c_y}\,c_x} \\ t[y] & \frac{-\theta\,c_x+e^{c_y}\,\theta\,c_x-\theta\,c_y+e^{c_y}\,\theta\,c_y}{-c_y+e^{c_x+c_y}\,c_y} & \frac{-\alpha\,c_x+e^{c_y}\,\alpha\,c_x-\alpha\,c_y+e^{c_y}\,\alpha\,c_y}{-c_y+e^{c_x+c_y}\,c_y} & \frac{-\alpha\,c_x+e^{c_y}\,\alpha\,c_x-\alpha\,c_y+e^{c_y}\,\alpha\,c_y}{-c_y+e^{c_x+c_y}\,c_y} \end{pmatrix}$$

Γ-Calculus

```
{Xp_ab, Xm_ab} // Γ
```

$$\left\{\begin{pmatrix} 1 & s_a & s_b \\ s_a & 1 & 1-T_a \\ s_b & 0 & T_a \\ \Sigma & 1 & T_a \end{pmatrix}, \begin{pmatrix} 1 & s_a & s_b \\ s_a & 1 & \frac{-1+T_a}{T_a} \\ s_b & 0 & \frac{1}{T_a} \\ \Sigma & 1 & \frac{1}{T_a} \end{pmatrix}\right\}$$

Meta-Associativity

$$n = 4;\ \gamma 0 = \Gamma\left[\omega,\ \sum_{a=0}^{n} h_a\,\sigma_a,\ \sum_{a=1}^{n}\sum_{b=1}^{n} t_a\, h_b\, \alpha_{10\,a+b}\right]$$

$$\begin{pmatrix} \omega & s_1 & s_2 & s_3 & s_4 \\ s_1 & \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ s_2 & \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ s_3 & \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\ s_4 & \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \\ \Sigma & \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 \end{pmatrix}$$

```
γ0 // dm[1, 2, 1] // dm[1, 3, 1]
```

$$\begin{pmatrix} \omega\,(1-\alpha_{12}-\alpha_{13}\,\alpha_{22}-\alpha_{23}+\alpha_{12}\,\alpha_{23}) & s_1 \\ s_1 & \frac{\alpha_{31}-\alpha_{12}\,\alpha_{31}-\alpha_{13}\,\alpha_{22}\,\alpha_{31}-\alpha_{23}\,\alpha_{31}+\alpha_{12}\,\alpha_{23}\,\alpha_{31}+\alpha_{11}\,\alpha_{32}+\alpha_{13}\,\alpha_{21}\,\alpha_{32}-\alpha_{11}\,\alpha_{23}\,\alpha_{32}+\alpha_{21}\,\alpha_{33}-\alpha_{12}\,\alpha_{21}\,\alpha_{33}+}{1-\alpha_{12}-\alpha_{13}\,\alpha_{22}-\alpha_{23}+\alpha_{12}\,\alpha_{23}} \\ s_4 & \frac{\alpha_{41}-\alpha_{12}\,\alpha_{41}-\alpha_{13}\,\alpha_{22}\,\alpha_{41}-\alpha_{23}\,\alpha_{41}+\alpha_{12}\,\alpha_{23}\,\alpha_{41}+\alpha_{11}\,\alpha_{42}+\alpha_{13}\,\alpha_{21}\,\alpha_{42}-\alpha_{11}\,\alpha_{23}\,\alpha_{42}+\alpha_{21}\,\alpha_{43}-\alpha_{12}\,\alpha_{21}\,\alpha_{43}+}{1-\alpha_{12}-\alpha_{13}\,\alpha_{22}-\alpha_{23}+\alpha_{12}\,\alpha_{23}} \\ \Sigma & \sigma_1\,\sigma_2\,\sigma_3 \end{pmatrix}$$

```
γ0 // dm[2, 3, 2] // dm[1, 2, 1]
```

$$\begin{pmatrix} \omega\,(1-\alpha_{12}-\alpha_{13}\,\alpha_{22}-\alpha_{23}+\alpha_{12}\,\alpha_{23}) & s_1 \\ s_1 & \frac{\alpha_{31}-\alpha_{12}\,\alpha_{31}-\alpha_{13}\,\alpha_{22}\,\alpha_{31}-\alpha_{23}\,\alpha_{31}+\alpha_{12}\,\alpha_{23}\,\alpha_{31}+\alpha_{11}\,\alpha_{32}+\alpha_{13}\,\alpha_{21}\,\alpha_{32}-\alpha_{11}\,\alpha_{23}\,\alpha_{32}+\alpha_{21}\,\alpha_{33}-\alpha_{12}\,\alpha_{21}\,\alpha_{33}+}{1-\alpha_{12}-\alpha_{13}\,\alpha_{22}-\alpha_{23}+\alpha_{12}\,\alpha_{23}} \\ s_4 & \frac{\alpha_{41}-\alpha_{12}\,\alpha_{41}-\alpha_{13}\,\alpha_{22}\,\alpha_{41}-\alpha_{23}\,\alpha_{41}+\alpha_{12}\,\alpha_{23}\,\alpha_{41}+\alpha_{11}\,\alpha_{42}+\alpha_{13}\,\alpha_{21}\,\alpha_{42}-\alpha_{11}\,\alpha_{23}\,\alpha_{42}+\alpha_{21}\,\alpha_{43}-\alpha_{12}\,\alpha_{21}\,\alpha_{43}+}{1-\alpha_{12}-\alpha_{13}\,\alpha_{22}-\alpha_{23}+\alpha_{12}\,\alpha_{23}} \\ \Sigma & \sigma_1\,\sigma_2\,\sigma_3 \end{pmatrix}$$

```
(γ0 // dm[1, 2, 1] // dm[1, 3, 1]) == (γ0 // dm[2, 3, 2] // dm[1, 2, 1])
```

```
True
```

Column Sums

```
Clear[α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ, μ, ω];
```

$$\gamma 0 = \Gamma\left[\omega,\ h_a\,\sigma_a + h_b\,\sigma_b + h_S\,\sigma_S,\ \{t_a, t_b, t_S\}.\begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \Xi \end{pmatrix}.\{h_a, h_b, h_S\}\right];$$

```
γ0 // dm[a, b, c]
```

$$\begin{pmatrix} -(-1+\beta)\,\omega & s_c & s_S \\ s_c & \frac{-\gamma+\beta\,\gamma-\alpha\,\delta}{-1+\beta} & \frac{-\epsilon+\beta\,\epsilon-\delta\,\theta}{-1+\beta} \\ s_S & \frac{-\phi+\beta\,\phi-\alpha\,\psi}{-1+\beta} & \frac{-\Xi+\beta\,\Xi-\theta\,\psi}{-1+\beta} \\ \Sigma & \sigma_a\,\sigma_b & \sigma_S \end{pmatrix}$$

$$\{1, 1\}.\begin{pmatrix} \frac{-\gamma+\beta\,\gamma-\alpha\,\delta}{-1+\beta} & \frac{-\epsilon+\beta\,\epsilon-\delta\,\theta}{-1+\beta} \\ \frac{-\phi+\beta\,\phi-\alpha\,\psi}{-1+\beta} & \frac{-\Xi+\beta\,\Xi-\theta\,\psi}{-1+\beta} \end{pmatrix}\ \text{// Simplify}$$

$$\left\{\frac{(-1+\beta)\,\gamma+(-1+\beta)\,\phi-\alpha\,(\delta+\psi)}{-1+\beta},\ \frac{\epsilon-\beta\,\epsilon+\delta\,\theta+\Xi-\beta\,\Xi+\theta\,\psi}{1-\beta}\right\}$$

$$\{1, 1\}.\begin{pmatrix} \frac{-\gamma+\beta\,\gamma-\alpha\,\delta}{-1+\beta} & \frac{-\epsilon+\beta\,\epsilon-\delta\,\theta}{-1+\beta} \\ \frac{-\phi+\beta\,\phi-\alpha\,\psi}{-1+\beta} & \frac{-\Xi+\beta\,\Xi-\theta\,\psi}{-1+\beta} \end{pmatrix}\ \text{/. }\{\alpha \to s1-\gamma-\phi,\ \delta \to s2-\beta-\psi,\ \Xi \to s3-\theta-\epsilon\}\ \text{// Simplify}$$

$$\left\{\frac{s1\,(-s2+\beta)+(-1+s2)\,(\gamma+\phi)}{-1+\beta},\ \frac{s3\,(-1+\beta)+\theta-s2\,\theta}{-1+\beta}\right\}$$

$$\{1, 1\}.\begin{pmatrix} \frac{-\gamma+\beta\,\gamma-\alpha\,\delta}{-1+\beta} & \frac{-\epsilon+\beta\,\epsilon-\delta\,\theta}{-1+\beta} \\ \frac{-\phi+\beta\,\phi-\alpha\,\psi}{-1+\beta} & \frac{-\Xi+\beta\,\Xi-\theta\,\psi}{-1+\beta} \end{pmatrix}\ \text{/. }\{\alpha \to s1-\gamma-\phi,\ \delta \to s2-\beta-\psi,\ \Xi \to s3-\theta-\epsilon\}\ \text{/.}$$

```
  s1 | s2 | s3 → 1 // Simplify
```

{1, 1}

Tangle Concatenation; Γ-inversion

$$\mathbf{n = 3;}\ \Big\{\gamma 1 = \Gamma\Big[\omega 1,\ \sum_{a=0}^{n} h_a\,\sigma 1_a,\ \sum_{a=1}^{n}\sum_{b=1}^{n} t_a\,h_b\,\alpha_{10\,a+b}\Big],$$

$$\gamma 2 = \Gamma\Big[\omega 2,\ \sum_{a=0}^{n} h_a\,\sigma 2_a,\ \sum_{a=1}^{n}\sum_{b=1}^{n} t_a\,h_b\,\beta_{10\,a+b}\Big],$$

```
FullStitch[γ1, γ2], γ1 ** γ2, FullStitch[γ1, γ2] == γ1 ** γ2}
```

$$\left\{\begin{pmatrix} \omega 1 & s_1 & s_2 & s_3 \\ s_1 & \alpha_{11} & \alpha_{12} & \alpha_{13} \\ s_2 & \alpha_{21} & \alpha_{22} & \alpha_{23} \\ s_3 & \alpha_{31} & \alpha_{32} & \alpha_{33} \\ \Sigma & \sigma 1_1 & \sigma 1_2 & \sigma 1_3 \end{pmatrix},\ \begin{pmatrix} \omega 2 & s_1 & s_2 & s_3 \\ s_1 & \beta_{11} & \beta_{12} & \beta_{13} \\ s_2 & \beta_{21} & \beta_{22} & \beta_{23} \\ s_3 & \beta_{31} & \beta_{32} & \beta_{33} \\ \Sigma & \sigma 2_1 & \sigma 2_2 & \sigma 2_3 \end{pmatrix},\right.$$

$$\begin{pmatrix} \omega 1\,\omega 2 & s_1 & s_2 & s_3 \\ s_1 & \alpha_{11}\beta_{11}+\alpha_{21}\beta_{12}+\alpha_{31}\beta_{13} & \alpha_{12}\beta_{11}+\alpha_{22}\beta_{12}+\alpha_{32}\beta_{13} & \alpha_{13}\beta_{11}+\alpha_{23}\beta_{12}+\alpha_{33}\beta_{13} \\ s_2 & \alpha_{11}\beta_{21}+\alpha_{21}\beta_{22}+\alpha_{31}\beta_{23} & \alpha_{12}\beta_{21}+\alpha_{22}\beta_{22}+\alpha_{32}\beta_{23} & \alpha_{13}\beta_{21}+\alpha_{23}\beta_{22}+\alpha_{33}\beta_{23} \\ s_3 & \alpha_{11}\beta_{31}+\alpha_{21}\beta_{32}+\alpha_{31}\beta_{33} & \alpha_{12}\beta_{31}+\alpha_{22}\beta_{32}+\alpha_{32}\beta_{33} & \alpha_{13}\beta_{31}+\alpha_{23}\beta_{32}+\alpha_{33}\beta_{33} \\ \Sigma & \sigma 1_1\,\sigma 2_1 & \sigma 1_2\,\sigma 2_2 & \sigma 1_3\,\sigma 2_3 \end{pmatrix},$$

$$\left.\begin{pmatrix} \omega 1\,\omega 2 & s_1 & s_2 & s_3 \\ s_1 & \alpha_{11}\beta_{11}+\alpha_{21}\beta_{12}+\alpha_{31}\beta_{13} & \alpha_{12}\beta_{11}+\alpha_{22}\beta_{12}+\alpha_{32}\beta_{13} & \alpha_{13}\beta_{11}+\alpha_{23}\beta_{12}+\alpha_{33}\beta_{13} \\ s_2 & \alpha_{11}\beta_{21}+\alpha_{21}\beta_{22}+\alpha_{31}\beta_{23} & \alpha_{12}\beta_{21}+\alpha_{22}\beta_{22}+\alpha_{32}\beta_{23} & \alpha_{13}\beta_{21}+\alpha_{23}\beta_{22}+\alpha_{33}\beta_{23} \\ s_3 & \alpha_{11}\beta_{31}+\alpha_{21}\beta_{32}+\alpha_{31}\beta_{33} & \alpha_{12}\beta_{31}+\alpha_{22}\beta_{32}+\alpha_{32}\beta_{33} & \alpha_{13}\beta_{31}+\alpha_{23}\beta_{32}+\alpha_{33}\beta_{33} \\ \Sigma & \sigma 1_1\,\sigma 2_1 & \sigma 1_2\,\sigma 2_2 & \sigma 1_3\,\sigma 2_3 \end{pmatrix},\ \text{True}\right\}$$

$$\gamma 1^{-1}$$

$$\begin{pmatrix} \frac{1}{\omega 1} & s_1 & s_2 \\ s_1 & \frac{\alpha_{23}\alpha_{32}-\alpha_{22}\alpha_{33}}{\alpha_{13}\alpha_{22}\alpha_{31}-\alpha_{12}\alpha_{23}\alpha_{31}-\alpha_{13}\alpha_{21}\alpha_{32}+\alpha_{11}\alpha_{23}\alpha_{32}+\alpha_{12}\alpha_{21}\alpha_{33}-\alpha_{11}\alpha_{22}\alpha_{33}} & \frac{\alpha_{13}\alpha_{32}-\alpha_{12}\alpha_{33}}{-\alpha_{13}\alpha_{22}\alpha_{31}+\alpha_{12}\alpha_{23}\alpha_{31}+\alpha_{13}\alpha_{21}\alpha_{32}-\alpha_{11}\alpha_{23}\alpha_{32}-\alpha_{12}\alpha_{21}\alpha_{33}+\alpha_{11}\alpha_{2\cdots}} \\ s_2 & \frac{\alpha_{23}\alpha_{31}-\alpha_{21}\alpha_{33}}{-\alpha_{13}\alpha_{22}\alpha_{31}+\alpha_{12}\alpha_{23}\alpha_{31}+\alpha_{13}\alpha_{21}\alpha_{32}-\alpha_{11}\alpha_{23}\alpha_{32}-\alpha_{12}\alpha_{21}\alpha_{33}+\alpha_{11}\alpha_{22}\alpha_{33}} & \frac{\alpha_{13}\alpha_{31}-\alpha_{11}\alpha_{33}}{\alpha_{13}\alpha_{22}\alpha_{31}-\alpha_{12}\alpha_{23}\alpha_{31}-\alpha_{13}\alpha_{21}\alpha_{32}+\alpha_{11}\alpha_{23}\alpha_{32}+\alpha_{12}\alpha_{21}\alpha_{33}-\alpha_{11}\alpha_{22}\cdots} \\ s_3 & \frac{\alpha_{22}\alpha_{31}-\alpha_{21}\alpha_{32}}{\alpha_{13}\alpha_{22}\alpha_{31}-\alpha_{12}\alpha_{23}\alpha_{31}-\alpha_{13}\alpha_{21}\alpha_{32}+\alpha_{11}\alpha_{23}\alpha_{32}+\alpha_{12}\alpha_{21}\alpha_{33}-\alpha_{11}\alpha_{22}\alpha_{33}} & \frac{\alpha_{12}\alpha_{31}-\alpha_{11}\alpha_{32}}{-\alpha_{13}\alpha_{22}\alpha_{31}+\alpha_{12}\alpha_{23}\alpha_{31}+\alpha_{13}\alpha_{21}\alpha_{32}-\alpha_{11}\alpha_{23}\alpha_{32}-\alpha_{12}\alpha_{21}\alpha_{33}+\alpha_{11}\alpha_{2\cdots}} \\ \Sigma & \frac{1}{\sigma 1_1} & \frac{1}{\sigma 1_2} \end{pmatrix}$$

$$\gamma 1 ** \gamma 1^{-1}$$

$$\begin{pmatrix} 1 & s_1 & s_2 & s_3 \\ s_1 & 1 & 0 & 0 \\ s_2 & 0 & 1 & 0 \\ s_3 & 0 & 0 & 1 \\ \Sigma & 1 & 1 & 1 \end{pmatrix}$$

```
γ1 // dS[1] // dS[2] // dS[3]
```

$$\begin{pmatrix} -\frac{\omega 1\,(\alpha_{13}\,\alpha_{22}\,\alpha_{31}-\alpha_{12}\,\alpha_{23}\,\alpha_{31}-\alpha_{13}\,\alpha_{21}\,\alpha_{32}+\alpha_{11}\,\alpha_{23}\,\alpha_{32}+\alpha_{12}\,\alpha_{21}\,\alpha_{33}-\alpha_{11}\,\alpha_{22}\,\alpha_{33})}{\sigma 1_1\,\sigma 1_2\,\sigma 1_3} & s_1 \\ s_1 & \frac{\alpha_{23}\,\alpha_{32}-\alpha_{22}\,\alpha_{33}}{\alpha_{13}\,\alpha_{22}\,\alpha_{31}-\alpha_{12}\,\alpha_{23}\,\alpha_{31}-\alpha_{13}\,\alpha_{21}\,\alpha_{32}+\alpha_{11}\,\alpha_{23}\,\alpha_{32}+\alpha_{12}\,\alpha_{21}\,\alpha_{33}-\alpha_{11}\,\alpha_{2}} \\ s_2 & \frac{\alpha_{23}\,\alpha_{31}-\alpha_{21}\,\alpha_{33}}{-\alpha_{13}\,\alpha_{22}\,\alpha_{31}+\alpha_{12}\,\alpha_{23}\,\alpha_{31}+\alpha_{13}\,\alpha_{21}\,\alpha_{32}-\alpha_{11}\,\alpha_{23}\,\alpha_{32}-\alpha_{12}\,\alpha_{21}\,\alpha_{33}+\alpha_{11}\,\alpha_{2}} \\ s_3 & \frac{\alpha_{22}\,\alpha_{31}-\alpha_{21}\,\alpha_{32}}{\alpha_{13}\,\alpha_{22}\,\alpha_{31}-\alpha_{12}\,\alpha_{23}\,\alpha_{31}-\alpha_{13}\,\alpha_{21}\,\alpha_{32}+\alpha_{11}\,\alpha_{23}\,\alpha_{32}+\alpha_{12}\,\alpha_{21}\,\alpha_{33}-\alpha_{11}\,\alpha_{2}} \\ \Sigma & \frac{1}{\sigma 1_1} \end{pmatrix}$$

```
(γ1 // dS[1] // dS[2] // dS[3]) == γ1^-1 // Simplify
```

$$-\frac{1}{\sigma 1_1\,\sigma 1_2\,\sigma 1_3}\omega 1\,(\alpha_{13}\,(\alpha_{22}\,\alpha_{31}-\alpha_{21}\,\alpha_{32})+\alpha_{12}\,(-\alpha_{23}\,\alpha_{31}+\alpha_{21}\,\alpha_{33})+\alpha_{11}\,(\alpha_{23}\,\alpha_{32}-\alpha_{22}\,\alpha_{33}))=\frac{1}{\omega 1}$$

Other

R3

```
{Xm51 Xm62 Xp34 // Γ // dm[1, 4, 1] // dm[2, 5, 2] // dm[3, 6, 3],
  Xp61 Xm24 Xm35 // Γ // dm[1, 4, 1] // dm[2, 5, 2] // dm[3, 6, 3]}
```

R3

$$\left\{\begin{pmatrix} 1 & s_1 & s_2 & s_3 \\ s_1 & \frac{T_3}{T_2} & 0 & 0 \\ s_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0 \\ s_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1 \\ \Sigma & \frac{T_3}{T_2} & \frac{1}{T_3} & 1 \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 & s_3 \\ s_1 & \frac{T_3}{T_2} & 0 & 0 \\ s_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0 \\ s_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1 \\ \Sigma & \frac{T_3}{T_2} & \frac{1}{T_3} & 1 \end{pmatrix}\right\}$$

$\gamma = Xm_{12,1}\, Xm_{27}\, Xm_{83}\, Xm_{4,11}\, Xp_{16,5}\, Xp_{6,13}\, Xp_{14,9}\, Xp_{10,15}$ // Γ

1	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}	s_{11}	s_{12}	s_{13}	s_{14}	s_{15}	s_{16}
s_1	$\frac{1}{T_{12}}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
s_2	0	1	0	0	0	0	$\frac{-1+T_2}{T_2}$	0	0	0	0	0	0	0	0	0
s_3	0	0	$\frac{1}{T_8}$	0	0	0	0	0	0	0	0	0	0	0	0	0
s_4	0	0	0	1	0	0	0	0	0	0	$\frac{-1+T_4}{T_4}$	0	0	0	0	0
s_5	0	0	0	0	T_{16}	0	0	0	0	0	0	0	0	0	0	0
s_6	0	0	0	0	0	1	0	0	0	0	0	0	$1-T_6$	0	0	0
s_7	0	0	0	0	0	0	$\frac{1}{T_2}$	0	0	0	0	0	0	0	0	0
s_8	0	0	$\frac{-1+T_8}{T_8}$	0	0	0	0	1	0	0	0	0	0	0	0	0
s_9	0	0	0	0	0	0	0	0	T_{14}	0	0	0	0	0	0	0
s_{10}	0	0	0	0	0	0	0	0	0	1	0	0	0	0	$1-T_{10}$	0
s_{11}	0	0	0	0	0	0	0	0	0	0	$\frac{1}{T_4}$	0	0	0	0	0
s_{12}	$\frac{-1+T_{12}}{T_{12}}$	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
s_{13}	0	0	0	0	0	0	0	0	0	0	0	0	T_6	0	0	0
s_{14}	0	0	0	0	0	0	0	0	$1-T_{14}$	0	0	0	0	1	0	0
s_{15}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	T_{10}	0
s_{16}	0	0	0	0	$1-T_{16}$	0	0	0	0	0	0	0	0	0	0	1
Σ	$\frac{1}{T_{12}}$	1	$\frac{1}{T_8}$	1	T_{16}	1	$\frac{1}{T_2}$	1	T_{14}	1	$\frac{1}{T_4}$	1	T_6	1	T_{10}	1

Do[$\gamma = \gamma$ // $dm_{1k\to 1}$, {k, 2, 10}]; γ

$\frac{T_1^2+T_{16}-T_1 T_{16}}{T_1^2}$	s_1	s_{11}	s_{12}	s_{13}	s_{14}	s_{15}	s_{16}
s_1	$\frac{T_{14}\left(-T_1+T_1^2+T_{16}\right)}{T_{12}\left(T_1^2+T_{16}-T_1 T_{16}\right)}$	$\frac{(-1+T_1)\left(1-T_1+T_1^2\right) T_{14} T_{16}}{T_1\left(T_1^2+T_{16}-T_1 T_{16}\right)}$	0	$-\frac{(-1+T_1)\left(1-T_1+T_1^2\right) T_{14}}{T_1^2+T_{16}-T_1 T_{16}}$	0	$1-T_1$	0
s_{11}	0	$\frac{1}{T_1}$	0	0	0	0	0
s_{12}	$\frac{-1+T_{12}}{T_{12}}$	0	1	0	0	0	0
s_{13}	0	0	0	T_1	0	0	0
s_{14}	$-\frac{(-1+T_{14})\left(-T_1+T_1^2+T_{16}\right)}{T_{12}\left(T_1^2+T_{16}-T_1 T_{16}\right)}$	$-\frac{(-1+T_1)\left(1-T_1+T_1^2\right)(-1+T_{14}) T_{16}}{T_1\left(T_1^2+T_{16}-T_1 T_{16}\right)}$	0	$\frac{(-1+T_1)\left(1-T_1+T_1^2\right)(-1+T_{14})}{T_1^2+T_{16}-T_1 T_{16}}$	1	0	0
s_{15}	0	0	0	0	0	T_1	0
s_{16}	$-\frac{T_1(-1+T_{16})}{T_{12}\left(T_1^2+T_{16}-T_1 T_{16}\right)}$	$-\frac{(-1+T_1) T_1(-1+T_{16})}{T_1^2+T_{16}-T_1 T_{16}}$	0	$\frac{(-1+T_1)^2(-1+T_{16})}{T_1^2+T_{16}-T_1 T_{16}}$	0	0	1
Σ	$\frac{T_{14} T_{16}}{T_1^2 T_{12}}$	$\frac{1}{T_1}$	1	T_1	1	T_1	1

8_17

$\gamma = Xm_{12,1}\, Xm_{27}\, Xm_{83}\, Xm_{4,11}\, Xp_{16,5}\, Xp_{6,13}\, Xp_{14,9}\, Xp_{10,15}$ // Γ;
Do[$\gamma = \gamma$ // dm[1, k, 1], {k, 2, 16}]; γ

8_17

$-\frac{1-4 T_1+8 T_1^2-11 T_1^3+8 T_1^4-4 T_1^5+T_1^6}{T_1^3}$	s_1
s_1	1
Σ	1

```
Z[Γ, Link["L6a4"]]
```

KnotTheory::loading : Loading precomputed data in PD4Links`.

$$\begin{pmatrix} \frac{(1-T_1-T_5+T_1 T_5-T_9+T_1 T_9+T_5 T_9)\,(T_1+T_5-T_1 T_5+T_9-T_1 T_9-T_5 T_9+T_1 T_5 T_9)}{T_1 T_5 T_9} & S_1 \\ S_1 & \frac{T_9\left(1-2 T_1+T_1^2-T_5+3 T_1 T_5-T_1^2 T_5-T_9+2 T_1 T_9-T_1^2 T_9+T_5 T_9-2 T_1 T_5 T_9+\cdots\right.}{(1-T_1-T_5+T_1 T_5-T_9+T_1 T_9+T_5 T_9)\,(T_1+T_5-T_1 T_5+T_9-T_1 T_9-T_5 T_9+T_1\cdots} \\ S_5 & \frac{T_1\,(-1+T_5)\,(1-T_1+T_1 T_5)\,(-1+T_9)}{(1-T_1-T_5+T_1 T_5-T_9+T_1 T_9+T_5 T_9)\,(T_1+T_5-T_1 T_5+T_9-T_1 T_9-T_5 T_9+T_1\cdots} \\ S_9 & \frac{(-1+T_5)\,T_5\,(-1+T_9)\,(1-2 T_1-T_9+T_1 T_9)}{(1-T_1-T_5+T_1 T_5-T_9+T_1 T_9+T_5 T_9)\,(T_1+T_5-T_1 T_5+T_9-T_1 T_9-T_5 T_9+T_1\cdots} \\ \Sigma & 1 \end{pmatrix}$$

```
MVA[Γ, Link["L6a4"]]
```

$$-\frac{(-1+T_1)\,(-1+T_5)\,(-1+T_9)}{T_1 T_5}$$

```
Factor[1/MVA[Γ, #] (MultivariableAlexander[#][T] /. T[i_] :→ T_Skeleton[#][[i,1]])] & /@
 AllLinks[{2, 8}]
```

KnotTheory::loading : Loading precomputed data in MultivariableAlexander4Links`.

$$\left\{-T_1^2 T_3,\ -T_1^{3/2} T_5^{3/2},\ -\sqrt{T_1}\, T_5^{3/2},\ -T_1^{3/2}\sqrt{T_5},\ -T_1^2 T_7^2,\ -T_1^2 T_7^2,\ -\frac{\sqrt{T_1}\sqrt{T_5}}{\sqrt{T_9}},\ -T_1^{3/2} T_5^{3/2} T_9^{3/2},\right.$$
$$-\frac{\sqrt{T_1}\sqrt{T_5}}{T_9^{3/2}},\ -\sqrt{T_1}\sqrt{T_5},\ -T_1^{3/2} T_5^{7/2},\ -\frac{\sqrt{T_1}}{T_5^{3/2}},\ -\frac{\sqrt{T_1}}{T_5^{3/2}},\ -T_1 T_7^2,\ -\frac{1}{T_7},\ -\frac{T_1^{3/2}\sqrt{T_5}}{\sqrt{T_9}},\ -T_1^{3/2} T_5^{7/2},$$
$$-\sqrt{T_1}\, T_5^{5/2},\ -\sqrt{T_1}\, T_5^{3/2},\ -\frac{\sqrt{T_1}}{\sqrt{T_5}},\ -T_1^{3/2} T_5^{3/2},\ -\sqrt{T_1}\, T_5^{3/2},\ -\frac{T_1^{3/2}}{\sqrt{T_5}},\ -\frac{T_1^{3/2}}{\sqrt{T_5}},\ -T_1^{3/2} T_5^{7/2},\ -\frac{T_1}{T_7},$$
$$-T_1 T_7,\ -T_1^2 T_7^3,\ -T_1^2 T_7^3,\ -T_1^{5/2} T_9^{5/2},\ -T_1^{5/2} T_9^{5/2},\ -T_1^{5/2} T_9^{5/2},\ -T_1^{3/2} T_5^{3/2}\sqrt{T_9},\ -\sqrt{T_1},\ -T_1^{3/2} T_5^2 T_{11}^2,$$
$$-\frac{\sqrt{T_1}}{T_{11}^2},\ -\frac{\sqrt{T_1}\, T_5}{T_{11}},\ -\frac{T_1^{3/2}\sqrt{T_5}}{T_{13}^{3/2}},\ -T_1^{3/2} T_5^{3/2} T_9^{3/2} T_{13}^{3/2},\ -T_1^{3/2} T_5^{3/2},\ -\sqrt{T_1}\sqrt{T_5},\ -T_1^{3/2} T_5^2 T_{11}^2,$$
$$\left.-\sqrt{T_1}\, T_5^2 T_{11},\ -\sqrt{T_1}\, T_5^{3/2} T_{11}^{3/2},\ -T_1^{3/2} T_5^{5/2}\sqrt{T_{13}},\ -\frac{\sqrt{T_1}\sqrt{T_5}}{\sqrt{T_9}\sqrt{T_{13}}},\ -\sqrt{T_1}\sqrt{T_5}\sqrt{T_9}\sqrt{T_{13}}\right\}$$

$\alpha\leftrightarrow\Gamma$ Convertions

```
{Xp[1, 2] // Γ, Xp[1, 2] // A // Γ}
```

$$\left\{\begin{pmatrix} 1 & S_1 & S_2 \\ S_1 & 1 & 1-T_1 \\ S_2 & 0 & T_1 \\ \Sigma & 1 & T_1 \end{pmatrix},\ \begin{pmatrix} 1 & S_1 & S_2 \\ S_1 & 1 & 1-T_1 \\ S_2 & 0 & T_1 \\ \Sigma & 1 & T_1 \end{pmatrix}\right\}$$

```
{Xm[1, 2] // A, Xm[1, 2] // Γ // A}
```

$$\left\{\begin{pmatrix} 1 & h[2] \\ t[1] & \frac{e^{-c_1}\,(1-e^{c_1})}{c_1} \end{pmatrix}, \begin{pmatrix} 1 & h[2] \\ t[1] & \frac{e^{-c_1}\,(1-e^{c_1})}{c_1} \end{pmatrix}\right\}$$

```
Clear[α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ, μ, ω];
```

$$\gamma 0 = \Gamma\left[\omega,\ h_a\,\sigma_a + h_b\,\sigma_b + h_S\,\sigma_S,\ \{t_a, t_b, t_S\}.\begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \Xi \end{pmatrix}.\{h_a, h_b, h_S\}\right];$$

```
{γ0, γ0 // A, γ0 // A // Γ, (γ0 // A // Γ) /. {α → 1 - γ - ϕ, δ → 1 - β - ψ, Ξ → 1 - θ - ϵ}}
```

$$\left\{\begin{pmatrix} \omega & s_a & s_b & s_S \\ s_a & \alpha & \beta & \theta \\ s_b & \gamma & \delta & \epsilon \\ s_S & \phi & \psi & \Xi \\ \Sigma & \sigma_a & \sigma_b & \sigma_S \end{pmatrix}, \begin{pmatrix} \omega & h[a] & h[b] & h[S] \\ t[a] & \frac{-\alpha+\sigma_a}{c_a} & -\frac{\beta}{c_a} & -\frac{\theta}{c_a} \\ t[b] & -\frac{\gamma}{c_b} & \frac{-\delta+\sigma_b}{c_b} & -\frac{\epsilon}{c_b} \\ t[S] & -\frac{\phi}{c_S} & -\frac{\psi}{c_S} & \frac{-\Xi+\sigma_S}{c_S} \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} \omega & s_a & s_b & s_S \\ s_a & 1-\gamma-\phi & \beta & \theta \\ s_b & \gamma & 1-\beta-\psi & \epsilon \\ s_S & \phi & \psi & 1-\epsilon-\theta \\ \Sigma & 1-\alpha-\gamma-\phi+\sigma_a & 1-\beta-\delta-\psi+\sigma_b & 1-\epsilon-\theta-\Xi+\sigma_S \end{pmatrix}, \begin{pmatrix} \omega & s_a & s_b & s_S \\ s_a & 1-\gamma-\phi & \beta & \theta \\ s_b & \gamma & 1-\beta-\psi & \epsilon \\ s_S & \phi & \psi & 1-\epsilon-\theta \\ \Sigma & \sigma_a & \sigma_b & \sigma_S \end{pmatrix}\right\}$$

```
Clear[α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ, μ, ω];
```

$$\gamma 0 = \Gamma\left[\omega,\ h_a\,\sigma_a + h_b\,\sigma_b + h_S\,\sigma_S,\ \{t_a, t_b, t_S\}.\begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \Xi \end{pmatrix}.\{h_a, h_b, h_S\}\right];$$

```
{γ0 // dm[a, b, c] // A, γ0 // A // dm[a, b, c]} /.
  {α → 1 - γ - ϕ, δ → 1 - β - ψ, Ξ → 1 - θ - ϵ}
```

$$\left\{\begin{pmatrix} \omega-\beta\,\omega & h[c] & h[S] \\ t[c] & \frac{1-\beta-\phi+\beta\,\phi-\psi+\gamma\,\psi+\phi\,\psi-\sigma_a\,\sigma_b+\beta\,\sigma_a\,\sigma_b}{-c_c+\beta\,c_c} & \frac{\epsilon-\beta\,\epsilon+\theta-\beta\,\theta-\theta\,\psi}{-c_c+\beta\,c_c} \\ t[S] & \frac{\phi-\beta\,\phi+\psi-\gamma\,\psi-\phi\,\psi}{-c_S+\beta\,c_S} & \frac{1-\beta-\epsilon+\beta\,\epsilon-\theta+\beta\,\theta+\theta\,\psi-\sigma_S+\beta\,\sigma_S}{-c_S+\beta\,c_S} \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} \omega-\beta\,\omega & h[c] & h[S] \\ t[c] & \frac{1-\beta-\phi+\beta\,\phi-\psi+\gamma\,\psi+\phi\,\psi-\sigma_a\,\sigma_b+\beta\,\sigma_a\,\sigma_b}{-c_c+\beta\,c_c} & \frac{\epsilon-\beta\,\epsilon+\theta-\beta\,\theta-\theta\,\psi}{-c_c+\beta\,c_c} \\ t[S] & \frac{\phi-\beta\,\phi+\psi-\gamma\,\psi-\phi\,\psi}{-c_S+\beta\,c_S} & \frac{1-\beta-\epsilon+\beta\,\epsilon-\theta+\beta\,\theta+\theta\,\psi-\sigma_S+\beta\,\sigma_S}{-c_S+\beta\,c_S} \end{pmatrix}\right\}$$

The KV solution in Γ, starting from α

```
V // A // αCollect[FullSimplify]
```

$$\begin{pmatrix} \frac{2^{1/4}\left(\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}\right)^{1/4}\left(\frac{\mathrm{Sinh}\left[\frac{c_2}{2}\right]}{c_2}\right)^{1/4}}{\left(\frac{\mathrm{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}\right)^{1/4}} & h[1] & \\ t[1] & \frac{-\sqrt{2}\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_2}{2}\right]}{c_2}}\,c_2+2\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_2\sqrt{\frac{\mathrm{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}}{2\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_1^2\sqrt{\frac{\mathrm{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}+2\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_1 c_2\sqrt{\frac{\mathrm{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}} & \\ t[2] & \frac{\sqrt{2}\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_2}{2}\right]}{c_2}}-2\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\sqrt{\frac{\mathrm{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}}{2\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_1\sqrt{\frac{\mathrm{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}+2\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_2\sqrt{\frac{\mathrm{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}} & \frac{-e^{\frac{c_1}{2}}c_1\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_2}{2}\right]}{c_2}}+e^{\frac{3c_1}{2}+c_2}c_1\sqrt{\ldots}}{\ldots} \end{pmatrix}$$

```
V // A // Γ //
 ΓCollect[Assuming[T₁ > 0 && T₂ > 0, (# /. {Sinh[x_] :> (ⅇ^x - ⅇ^-x)/2}) // FullSimplify] &]
```

$$\begin{pmatrix} \frac{\left(\frac{-1+T_1}{\mathrm{Log}[T_1]}\right)^{1/4}\left(\frac{-1+T_2}{\mathrm{Log}[T_2]}\right)^{1/4}}{\left(\frac{-1+T_1T_2}{\mathrm{Log}[T_1T_2]}\right)^{1/4}} & s_1 & s_2 \\ s_1 & \frac{\mathrm{Log}[T_1]\left(\mathrm{Log}[T_2]\sqrt{\frac{(-1+T_1)T_1}{\mathrm{Log}[T_1]}}\sqrt{\frac{-1+T_2}{\mathrm{Log}[T_2]}}-\sqrt{\frac{-1+T_1T_2}{\mathrm{Log}[T_1T_2]}}+T_1\sqrt{\frac{-1+T_1T_2}{\mathrm{Log}[T_1T_2]}}\right)}{\mathrm{Log}[T_1T_2](-1+T_1)\sqrt{\frac{-1+T_1T_2}{\mathrm{Log}[T_1T_2]}}} & -\frac{\mathrm{Log}[T_1]\left(\sqrt{\frac{(-1+T_1)T_1}{\mathrm{Log}[T_1]}}\,T_2-\sqrt{\frac{-1+T_2}{\mathrm{Log}[T_2]}}\ldots\right.}{\mathrm{Log}[T_1T_2]\sqrt{\frac{-1+T_2}{\mathrm{Log}[T_2]}}\sqrt{\ldots}} \\ s_2 & \frac{\mathrm{Log}[T_2]\left(-T_1\sqrt{\frac{-1+T_2}{\mathrm{Log}[T_2]}}+\sqrt{\frac{(-1+T_1)T_1}{\mathrm{Log}[T_1]}}\sqrt{\frac{-1+T_1T_2}{\mathrm{Log}[T_1T_2]}}\right)}{\mathrm{Log}[T_1T_2]\sqrt{\frac{(-1+T_1)T_1}{\mathrm{Log}[T_1]}}\sqrt{\frac{-1+T_1T_2}{\mathrm{Log}[T_1T_2]}}} & \frac{\mathrm{Log}[T_1]\sqrt{\frac{(-1+T_1)T_1}{\mathrm{Log}[T_1]}}\,T_2+\mathrm{Log}[T_2]\sqrt{\ldots}}{\mathrm{Log}[T_1T_2]\sqrt{\frac{-1+T_2}{\mathrm{Log}[T_2]}}\sqrt{\ldots}} \\ \Sigma & 1 & \sqrt{T_1} \end{pmatrix}$$

```
ΓSimp = Assuming[T₁ > 1 && T₂ > 1, (# /. {Sinh[x_] :> (ⅇ^x - ⅇ^-x)/2}) // FullSimplify] &;
V // A // Γ
```

$$\begin{pmatrix} \left(\frac{\mathrm{Log}[T_1T_2](-1+T_1)(-1+T_2)}{\mathrm{Log}[T_1]\,\mathrm{Log}[T_2](-1+T_1T_2)}\right)^{1/4} & s_1 & s_2 \\ s_1 & \frac{\mathrm{Log}[T_1]+\sqrt{\frac{\mathrm{Log}[T_1]\mathrm{Log}[T_2]\mathrm{Log}[T_1T_2]T_1(-1+T_2)}{(-1+T_1)(-1+T_1T_2)}}}{\mathrm{Log}[T_1T_2]} & \frac{\mathrm{Log}[T_1]\left(-1+T_2\left(T_1-\sqrt{\frac{\mathrm{Log}[T_2]\mathrm{Log}[T_1T\ldots}{\mathrm{Log}\ldots}}\right.\right.}{\mathrm{Log}[T_1T_2](-1+T_1\ldots} \\ s_2 & -\frac{\mathrm{Log}[T_2]\left(-1+\sqrt{\frac{\mathrm{Log}[T_1]\mathrm{Log}[T_1T_2]T_1(-1+T_2)}{\mathrm{Log}[T_2](-1+T_1)(-1+T_1T_2)}}\right)}{\mathrm{Log}[T_1T_2]} & \frac{-\mathrm{Log}[T_2]+T_2\left(\mathrm{Log}[T_2]T_1+\sqrt{\frac{\mathrm{Log}[T_1]^3\mathrm{Log}[T_2](-1+T_1)T_1(-1+T\ldots}{\mathrm{Log}[T_1T_2](-1+T_2)}}\right.}{\mathrm{Log}[T_1T_2](-1+T_1\ldots} \\ \Sigma & 1 & \sqrt{T_1} \end{pmatrix}$$

```
Γ[V] ** Γ[Vi]
```

$$\begin{pmatrix} 1 & s_1 \\ s_1 & \frac{-\text{Log}[T_1 T_2]+T_2\left(\sqrt{\frac{\text{Log}[T_1]\text{Log}[T_2]\text{Log}[T_1 T_2](-1+T_1)T_1(-1+T_2)}{-1+T_1 T_2}}+T_1\left(\text{Log}[T_1 T_2]+\sqrt{\frac{\text{Log}[T_1]\text{Log}[T_2]\text{Log}[T_1 T_2](-1+T_2)}{(-1+T_1)T_1(-1+T_1 T_2)}}-T_1\sqrt{\frac{\text{Log}[T_1]\text{Log}[T_2]\text{Log}[T_1 T_2](-1+\cdots}{(-1+T_1)T_1(-1+T_1 T_2)}}\right.\right.}{\text{Log}[T_1 T_2](-1+T_1 T_2)} \\ s_2 & \frac{(-1+T_2)\left(-(-1+T_1)T_1\sqrt{\frac{\text{Log}[T_1]\text{Log}[T_2]\text{Log}[T_1 T_2](-1+T_2)}{(-1+T_1)T_1(-1+T_1 T_2)}}+\sqrt{\frac{\text{Log}[T_1]\text{Log}[T_2]\text{Log}[T_1 T_2](-1+T_1)T_1(-1+T_2)}{-1+T_1 T_2}}\right)}{\text{Log}[T_1 T_2](-1+T_1)(-1+T_1 T_2)} \\ \Sigma & 1 \end{pmatrix}$$

$$\frac{1}{\text{Log}[T_1 T_2](-1+T_1 T_2)}\left(-\text{Log}[T_1 T_2]+T_2\left(\sqrt{\frac{\text{Log}[T_1]\text{Log}[T_2]\text{Log}[T_1 T_2](-1+T_1)T_1(-1+T_2)}{-1+T_1 T_2}}+T_1\left(\text{Log}[T_1 T_2]+\sqrt{\frac{\text{Log}[T_1]\text{Log}[T_2]\text{Log}[T_1 T_2](-1+T_2)}{(-1+T_1)T_1(-1+T_1 T_2)}}-T_1\sqrt{\frac{\text{Log}[T_1]\text{Log}[T_2]\text{Log}[T_1 T_2](-1+T_2)}{(-1+T_1)T_1(-1+T_1 T_2)}}\right)\right)\right) \text{ // PowerExpand // Simplify}$$

1

$$\left((-1+T_2)\left(-(-1+T_1)T_1\sqrt{\frac{\text{Log}[T_1]\text{Log}[T_2]\text{Log}[T_1 T_2](-1+T_2)}{(-1+T_1)T_1(-1+T_1 T_2)}}+\sqrt{\frac{\text{Log}[T_1]\text{Log}[T_2]\text{Log}[T_1 T_2](-1+T_1)T_1(-1+T_2)}{-1+T_1 T_2}}\right)\right)\Big/\left(\text{Log}[T_1 T_2](-1+T_1)(-1+T_1 T_2)\right) \text{ // PowerExpand // Simplify}$$

0

dS and dA for Γ, starting from α

```
Clear[α, θ, ϕ, Ξ, ω];
γ0 = Γ[ω, h_a σ_a + h_S σ_S, {t_a, t_S}.{{α, θ}, {ϕ, Ξ}}.{h_a, h_S}];
((γ0 // A // dS[a] // Γ) == (γ0 // A // dA[a] // Γ)) /. {α → 1 - ϕ, Ξ → 1 - θ}
True
```

```
(γ0 // A // dS[a] // Γ)
```

$$\begin{pmatrix} \frac{(-1+\phi)\,\omega}{-1+\alpha+\phi-\sigma_a} & s_a & s_S \\ s_a & -\frac{1}{-1+\phi} & -\frac{\theta}{-1+\phi} \\ s_S & \frac{\phi}{-1+\phi} & \frac{-1+\theta+\phi}{-1+\phi} \\ \Sigma & -\frac{1}{-1+\alpha+\phi-\sigma_a} & -\frac{1-\alpha-\theta+\alpha\,\theta-\Xi+\alpha\,\Xi-\phi+\theta\,\phi+\Xi\,\phi+\sigma_a-\theta\,\sigma_a-\Xi\,\sigma_a+\sigma_S-\alpha\,\sigma_S-\phi\,\sigma_S+\sigma_a\,\sigma_S}{-1+\alpha+\phi-\sigma_a} \end{pmatrix}$$

```
(γ0 // A // dA[a] // Γ) /. {ϕ → 1 - α, Ξ → 1 - θ} // ΓCollect
```

$$\begin{pmatrix} \frac{\alpha\,\omega}{\sigma_a} & s_a & s_S \\ s_a & \frac{1}{\alpha} & \frac{\theta}{\alpha} \\ s_S & \frac{-1+\alpha}{\alpha} & \frac{\alpha-\theta}{\alpha} \\ \Sigma & \frac{1}{\sigma_a} & \sigma_S \end{pmatrix}$$

```
(γ0 // dA[a]) /. {ϕ → 1 - α, Ξ → 1 - θ} // ΓCollect
```

$$\begin{pmatrix} \frac{\alpha\,\omega}{\sigma_a} & s_a & s_S \\ s_a & \frac{1}{\alpha} & \frac{\theta}{\alpha} \\ s_S & \frac{-1+\alpha}{\alpha} & \frac{\alpha-\theta}{\alpha} \\ \Sigma & \frac{1}{\sigma_a} & \sigma_S \end{pmatrix}$$

```
(γ0 // A // dA[a] // Γ) == (γ0 // dA[a]) /. {ϕ → 1 - α, Ξ → 1 - θ} // Simplify
True
```

dΔ for Γ, starting from α

```
Clear[α, θ, ϕ, Ξ, ω];
```

$$\gamma 0 = \Gamma\left[\omega,\ h_a\,\sigma_a + h_S\,\sigma_S,\ \{t_a, t_S\}.\begin{pmatrix} \alpha & \theta \\ \phi & \Xi \end{pmatrix}.\{h_a, h_S\}\right];$$

```
((γ0 // A // dΔ[a, b, c] // Γ) /. {α → 1 - ϕ, Ξ → 1 - θ}) // ΓCollect
```

$$\begin{pmatrix} \omega & s_b & s_c & s_S \\ s_b & -\frac{-\text{Log}[T_b]+\phi\,\text{Log}[T_b]-\text{Log}[T_c]\,\sigma_a}{\text{Log}[T_b]+\text{Log}[T_c]} & -\frac{\text{Log}[T_b]\,(-1+\phi+\sigma_a)}{\text{Log}[T_b]+\text{Log}[T_c]} & \frac{\theta\,\text{Log}[T_b]}{\text{Log}[T_b]+\text{Log}[T_c]} \\ s_c & -\frac{\text{Log}[T_c]\,(-1+\phi+\sigma_a)}{\text{Log}[T_b]+\text{Log}[T_c]} & \frac{\text{Log}[T_c]-\phi\,\text{Log}[T_c]+\text{Log}[T_b]\,\sigma_a}{\text{Log}[T_b]+\text{Log}[T_c]} & \frac{\theta\,\text{Log}[T_c]}{\text{Log}[T_b]+\text{Log}[T_c]} \\ s_S & \phi & \phi & 1-\theta \\ \Sigma & \sigma_a & \sigma_a & \sigma_S \end{pmatrix}$$

```
Clear[α, θ, ϕ, Ξ, ω];
γ0 = Γ[ω, h_a σ_a + h_S σ_S, {t_a, t_S}.(α θ; ϕ Ξ).{h_a, h_S}];
γ0 // A // dΔ[a, b, c] // dS[c] // dm[b, c, a]
```

Power::infy : Infinite expression $\frac{1}{0}$ encountered. ≫

Power::infy : Infinite expression $\frac{1}{0}$ encountered. ≫

Infinity::indet : Indeterminate expression ComplexInfinity + ComplexInfinity + $\frac{(\Xi - \theta\phi - \Xi\phi - \sigma_S + \phi\sigma_S)\,t[S]}{-c_S + \phi c_S}$ encountered. ≫

$\left(\frac{-\omega+\phi\,\omega}{-1+\alpha+\phi-\sigma_a}\right)$

qΔ for Γ, starting from α

```
Clear[α, θ, ϕ, Ξ, ω];
γ0 = Γ[ω, h_a σ_a + h_S σ_S, {t_a, t_S}.(α θ; ϕ Ξ).{h_a, h_S}];
γ0 // A // qΔ[a, b, c] // Γ
```

ω	S_b	S_c	
S_b	$-\frac{1-\alpha-\phi+\alpha T_c-T_b T_c+\phi T_b T_c+\sigma_a-T_c\sigma_a}{-1+T_b T_c}$	$\frac{(-1+T_b)\,T_c\,(\alpha-\sigma_a)}{-1+T_b T_c}$	$\frac{\theta\,(-}{-}$
S_c	$\frac{(-1+T_c)\,(\alpha-\sigma_a)}{-1+T_b T_c}$	$-\frac{1-\phi-\alpha T_c-T_b T_c+\alpha T_b T_c+\phi T_b T_c+T_c\sigma_a-T_b T_c\sigma_a}{-1+T_b T_c}$	$\frac{\theta}{-}$
S_S	ϕ	ϕ	
Σ	$1-\alpha-\phi+\sigma_a$	$1-\alpha-\phi+\sigma_a$	$1-\epsilon$

```
Clear[α, θ, ϕ, Ξ, ω];
γ0 = Γ[ω, h_a σ_a + h_S σ_S, {t_a, t_S}.(α θ; ϕ Ξ).{h_a, h_S}];
γ0 // qΔ[a, b, c]
```

ω	S_b	S_c	S_S
S_b	$\frac{-\alpha T_c+\alpha T_b T_c-\sigma_a+T_c\sigma_a}{-1+T_b T_c}$	$\frac{(-1+T_b)\,T_c\,(\alpha-\sigma_a)}{-1+T_b T_c}$	$\frac{\theta\,(-1+T_b)\,T_c}{-1+T_b T_c}$
S_c	$\frac{(-1+T_c)\,(\alpha-\sigma_a)}{-1+T_b T_c}$	$\frac{-\alpha+\alpha T_c-T_c\sigma_a+T_b T_c\sigma_a}{-1+T_b T_c}$	$\frac{\theta\,(-1+T_c)}{-1+T_b T_c}$
S_S	ϕ	ϕ	Ξ
Σ	σ_a	σ_a	σ_S

```
Clear[α, θ, ϕ, Ξ, ω];
γ0 = Γ[ω, h_a σ_a + h_S σ_S, {t_a, t_S}.(α θ; ϕ Ξ).{h_a, h_S}];
Simplify[
 (γ0 // qΔ[a, b, c])[[3]] == (γ0 // A // qΔ[a, b, c] // Γ)[[3]] /. {α → 1 - ϕ, θ → 1 - Ξ}]
True
```

qΔ tests for Γ

$$\{\mathbf{t1} = \mathbf{Xp}_{13} \mathbin{//} \Gamma \mathbin{//} \mathbf{q\Delta}[1, 1, 2],\ \mathbf{t2} = (\epsilon[1]\ \mathbf{Xp}_{23} \mathbin{//} \Gamma) ** (\epsilon[2]\ \mathbf{Xp}_{13} \mathbin{//} \Gamma),\ \mathbf{t1} == \mathbf{t2}\}$$

$$\left\{\begin{pmatrix} 1 & s_1 & s_2 & s_3 \\ s_1 & 1 & 0 & -(-1+T_1)\,T_2 \\ s_2 & 0 & 1 & 1-T_2 \\ s_3 & 0 & 0 & T_1 T_2 \\ \Sigma & 1 & 1 & T_1 T_2 \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 & s_3 \\ s_1 & 1 & 0 & -(-1+T_1)\,T_2 \\ s_2 & 0 & 1 & 1-T_2 \\ s_3 & 0 & 0 & T_1 T_2 \\ \Sigma & 1 & 1 & T_1 T_2 \end{pmatrix}, \text{True}\right\}$$

$$\{\mathbf{t1} = \mathbf{Xm}_{13} \mathbin{//} \Gamma \mathbin{//} \mathbf{q\Delta}[1, 1, 2],\ \mathbf{t2} = (\epsilon[2]\ \mathbf{Xm}_{13} \mathbin{//} \Gamma) ** (\epsilon[1]\ \mathbf{Xm}_{23} \mathbin{//} \Gamma),\ \mathbf{t1} == \mathbf{t2}\}$$

$$\left\{\begin{pmatrix} 1 & s_1 & s_2 & s_3 \\ s_1 & 1 & 0 & \frac{-1+T_1}{T_1} \\ s_2 & 0 & 1 & \frac{-1+T_2}{T_1 T_2} \\ s_3 & 0 & 0 & \frac{1}{T_1 T_2} \\ \Sigma & 1 & 1 & \frac{1}{T_1 T_2} \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 & s_3 \\ s_1 & 1 & 0 & \frac{-1+T_1}{T_1} \\ s_2 & 0 & 1 & \frac{-1+T_2}{T_1 T_2} \\ s_3 & 0 & 0 & \frac{1}{T_1 T_2} \\ \Sigma & 1 & 1 & \frac{1}{T_1 T_2} \end{pmatrix}, \text{True}\right\}$$

$$\{\mathbf{t1} = \mathbf{Xp}_{3,1} \mathbin{//} \Gamma \mathbin{//} \mathbf{q\Delta}[1, 1, 2],\ \mathbf{t2} = (\epsilon[1]\ \mathbf{Xp}_{3,2} \mathbin{//} \Gamma) ** (\epsilon[2]\ \mathbf{Xp}_{3,1} \mathbin{//} \Gamma),\ \mathbf{t1} == \mathbf{t2}\}$$

$$\left\{\begin{pmatrix} 1 & s_1 & s_2 & s_3 \\ s_1 & T_3 & 0 & 0 \\ s_2 & 0 & T_3 & 0 \\ s_3 & 1-T_3 & 1-T_3 & 1 \\ \Sigma & T_3 & T_3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 & s_3 \\ s_1 & T_3 & 0 & 0 \\ s_2 & 0 & T_3 & 0 \\ s_3 & 1-T_3 & 1-T_3 & 1 \\ \Sigma & T_3 & T_3 & 1 \end{pmatrix}, \text{True}\right\}$$

$$\{\mathbf{t1} = \mathbf{Xm}_{3,1} \mathbin{//} \Gamma \mathbin{//} \mathbf{q\Delta}[1, 1, 2],\ \mathbf{t2} = (\epsilon[2]\ \mathbf{Xm}_{3,1} \mathbin{//} \Gamma) ** (\epsilon[1]\ \mathbf{Xm}_{3,2} \mathbin{//} \Gamma),\ \mathbf{t1} == \mathbf{t2}\}$$

$$\left\{\begin{pmatrix} 1 & s_1 & s_2 & s_3 \\ s_1 & \frac{1}{T_3} & 0 & 0 \\ s_2 & 0 & \frac{1}{T_3} & 0 \\ s_3 & \frac{-1+T_3}{T_3} & \frac{-1+T_3}{T_3} & 1 \\ \Sigma & \frac{1}{T_3} & \frac{1}{T_3} & 1 \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 & s_3 \\ s_1 & \frac{1}{T_3} & 0 & 0 \\ s_2 & 0 & \frac{1}{T_3} & 0 \\ s_3 & \frac{-1+T_3}{T_3} & \frac{-1+T_3}{T_3} & 1 \\ \Sigma & \frac{1}{T_3} & \frac{1}{T_3} & 1 \end{pmatrix}, \text{True}\right\}$$

$$\mathbf{Clear}[\alpha, \beta, \gamma, \delta, \theta, \epsilon, \phi, \psi, \Xi, \mu, \omega];$$

$$\left\{\gamma 0 = \Gamma\left[\omega,\ h_a \sigma_a + h_b \sigma_b + h_S \sigma_S,\ \{t_a, t_b, t_S\}.\begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \Xi \end{pmatrix}.\{h_a, h_b, h_S\}\right],\ \gamma 0 \mathbin{//} \mathbf{dm}[a, b, c]\right\}$$

$$\left\{\begin{pmatrix} \omega & s_a & s_b & s_S \\ s_a & \alpha & \beta & \theta \\ s_b & \gamma & \delta & \epsilon \\ s_S & \phi & \psi & \Xi \\ \Sigma & \sigma_a & \sigma_b & \sigma_S \end{pmatrix}, \begin{pmatrix} -(-1+\beta)\,\omega & s_c & s_S \\ s_c & \frac{-\gamma+\beta\gamma-\alpha\delta}{-1+\beta} & \frac{-\epsilon+\beta\epsilon-\delta\theta}{-1+\beta} \\ s_S & \frac{-\phi+\beta\phi-\alpha\psi}{-1+\beta} & \frac{-\Xi+\beta\Xi-\theta\psi}{-1+\beta} \\ \Sigma & \sigma_a \sigma_b & \sigma_S \end{pmatrix}\right\}$$

```
γ0 // dm[a, b, c] // qΔ[c, c1, c2]
```

$$\begin{pmatrix} -(-1+\beta)\,\omega & s_{c1} & \\ s_{c1} & \frac{\gamma T_{c2}-\beta\gamma T_{c2}+\alpha\delta T_{c2}-\gamma T_{c1} T_{c2}+\beta\gamma T_{c1} T_{c2}-\alpha\delta T_{c1} T_{c2}+\sigma_a\sigma_b-\beta\sigma_a\sigma_b-T_{c2}\sigma_a\sigma_b+\beta T_{c2}\sigma_a\sigma_b}{(-1+\beta)(-1+T_{c1} T_{c2})} & \frac{(-1+T_{c1})\,T_{c}}{(-} \\ s_{c2} & \frac{(-1+T_{c2})(-\gamma+\beta\gamma-\alpha\delta+\sigma_a\sigma_b-\beta\sigma_a\sigma_b)}{(-1+\beta)(-1+T_{c1} T_{c2})} & \frac{\gamma-\beta\gamma+\alpha\delta-\gamma T_{c2}+\beta\gamma T_{c2}-\alpha\delta T_{c2}+}{(-} \\ s_S & \frac{-\phi+\beta\phi-\alpha\psi}{-1+\beta} & \\ \Sigma & \sigma_a\sigma_b & \end{pmatrix}$$

```
γ0 // qΔ[a, a1, a2] // qΔ[b, b1, b2] // dm[a1, b1, c1] // dm[a2, b2, c2]
```

$$\begin{pmatrix} -(-1+\beta)\,\omega & s_{c1} & \\ s_{c1} & \frac{\gamma T_{c2}-\beta\gamma T_{c2}+\alpha\delta T_{c2}-\gamma T_{c1} T_{c2}+\beta\gamma T_{c1} T_{c2}-\alpha\delta T_{c1} T_{c2}+\sigma_a\sigma_b-\beta\sigma_a\sigma_b-T_{c2}\sigma_a\sigma_b+\beta T_{c2}\sigma_a\sigma_b}{(-1+\beta)(-1+T_{c1} T_{c2})} & \frac{(-1+T_{c1})\,T_{c}}{(-} \\ s_{c2} & \frac{(-1+T_{c2})(-\gamma+\beta\gamma-\alpha\delta+\sigma_a\sigma_b-\beta\sigma_a\sigma_b)}{(-1+\beta)(-1+T_{c1} T_{c2})} & \frac{\gamma-\beta\gamma+\alpha\delta-\gamma T_{c2}+\beta\gamma T_{c2}-\alpha\delta T_{c2}+}{(-} \\ s_S & \frac{-\phi+\beta\phi-\alpha\psi}{-1+\beta} & \\ \Sigma & \sigma_a\sigma_b & \end{pmatrix}$$

```
(γ0 // dm[a, b, c] // qΔ[c, c1, c2]) ==
  (γ0 // qΔ[a, a1, a2] // qΔ[b, b1, b2] // dm[a1, b1, c1] //
    dm[a2, b2, c2]) // Simplify
```

True

dS tests for Γ

```
{Xp[1, 2] // Γ, Xm[1, 2] // Γ // dS[1], Xm[1, 2] // Γ // dS[2]}
```

$$\left\{\begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & 1-T_1 \\ s_2 & 0 & T_1 \\ \Sigma & 1 & T_1 \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & 1-T_1 \\ s_2 & 0 & T_1 \\ \Sigma & 1 & T_1 \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & 1-T_1 \\ s_2 & 0 & T_1 \\ \Sigma & 1 & T_1 \end{pmatrix}\right\}$$

```
{Xm[1, 2] // Γ, Xp[1, 2] // Γ // dS[1], Xp[1, 2] // Γ // dS[2]}
```

$$\left\{\begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & \frac{-1+T_1}{T_1} \\ s_2 & 0 & \frac{1}{T_1} \\ \Sigma & 1 & \frac{1}{T_1} \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & \frac{-1+T_1}{T_1} \\ s_2 & 0 & \frac{1}{T_1} \\ \Sigma & 1 & \frac{1}{T_1} \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & \frac{-1+T_1}{T_1} \\ s_2 & 0 & \frac{1}{T_1} \\ \Sigma & 1 & \frac{1}{T_1} \end{pmatrix}\right\}$$

```
Xp[1, 2] // Γ // dS[1] // dS[2]
```

$$\begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & 1-T_1 \\ s_2 & 0 & T_1 \\ \Sigma & 1 & T_1 \end{pmatrix}$$

```
Clear[α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ, μ, ω];
{γ0 = Γ[ω, h_a σ_a + h_b σ_b + h_S σ_S, {t_a, t_b, t_S}.(α β θ; γ δ ϵ; ϕ ψ Ξ).{h_a, h_b, h_S}],
 t1 = γ0 // dm[a, b, c] // dS[c], t2 = γ0 // dS[a] // dS[b] // dm[b, a, c], t1 == t2}
```

$$\left\{\begin{pmatrix} \omega & s_a & s_b & s_S \\ s_a & \alpha & \beta & \theta \\ s_b & \gamma & \delta & \epsilon \\ s_S & \phi & \psi & \Xi \\ \Sigma & \sigma_a & \sigma_b & \sigma_S \end{pmatrix}, \begin{pmatrix} -\frac{(-\gamma+\beta\gamma-\alpha\delta)\,\omega}{\sigma_a\sigma_b} & s_c & s_S \\ s_c & \frac{-1+\beta}{-\gamma+\beta\gamma-\alpha\delta} & \frac{-\epsilon+\beta\epsilon-\delta\theta}{-\gamma+\beta\gamma-\alpha\delta} \\ s_S & -\frac{-\phi+\beta\phi-\alpha\psi}{-\gamma+\beta\gamma-\alpha\delta} & \frac{-\gamma\Xi+\beta\gamma\Xi-\alpha\delta\Xi+\epsilon\phi-\beta\epsilon\phi+\delta\theta\phi+\alpha\epsilon\psi-\gamma\theta\psi}{-\gamma+\beta\gamma-\alpha\delta} \\ \Sigma & \frac{1}{\sigma_a\sigma_b} & \sigma_S \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} -\frac{(-\gamma+\beta\gamma-\alpha\delta)\,\omega}{\sigma_a\sigma_b} & s_c & s_S \\ s_c & \frac{-1+\beta}{-\gamma+\beta\gamma-\alpha\delta} & \frac{-\epsilon+\beta\epsilon-\delta\theta}{-\gamma+\beta\gamma-\alpha\delta} \\ s_S & -\frac{-\phi+\beta\phi-\alpha\psi}{-\gamma+\beta\gamma-\alpha\delta} & \frac{-\gamma\Xi+\beta\gamma\Xi-\alpha\delta\Xi+\epsilon\phi-\beta\epsilon\phi+\delta\theta\phi+\alpha\epsilon\psi-\gamma\theta\psi}{-\gamma+\beta\gamma-\alpha\delta} \\ \Sigma & \frac{1}{\sigma_a\sigma_b} & \sigma_S \end{pmatrix}, \text{True}\right\}$$

```
Clear[α, θ, ϕ, Ξ, ω];
γ0 = Γ[ω, h_a σ_a + h_S σ_S, {t_a, t_S}.(α θ; ϕ Ξ).{h_a, h_S}];
FullSimplify[{1, 1}.dS[a][γ0][A], And @@ Thread[{1, 1}.γ0[A] == {1, 1}]]
```

$$\left\{\frac{1-\phi}{\alpha}, \frac{1-\phi}{\alpha}\right\}$$

```
{1, 1}.dS[a][γ0][A] // Simplify
```

$$\left\{\frac{1-\phi}{\alpha}, \frac{\theta+\alpha\,\Xi-\theta\,\phi}{\alpha}\right\}$$

```
And @@ Thread[{1, 1}.γ0[A] == {1, 1}]
```

$\alpha + \phi == 1 \,\&\&\, \theta + \Xi == 1$

```
Clear[α, θ, ϕ, Ξ, ω];
γ0 = Γ[ω, h_a σ_a + h_S σ_S, {t_a, t_S}.(α θ; ϕ Ξ).{h_a, h_S}];
γ0 // dS[a] // dS[a]
```

$$\begin{pmatrix} \omega & s_a & s_S \\ s_a & \alpha & \theta \\ s_S & \phi & \Xi \\ \Sigma & \sigma_a & \sigma_S \end{pmatrix}$$

```
Clear[α, θ, ϕ, Ξ, ω];
γ0 = Γ[ω, h_a σ_a + h_S σ_S, {t_a, t_S}.( α[T_a]  θ[T_a] ; ϕ[T_a]  Ξ[T_a] ).{h_a, h_S}];
{(γ0 // dη[a]) (ϵ[a] // Γ), γ0 // qΔ[a, b, c] // dS[c] // dm[b, c, a],
 γ0 // qΔ[a, b, c] // dS[c] // dm[c, b, a],
 γ0 // qΔ[a, b, c] // dS[b] // dm[b, c, a],
 γ0 // qΔ[a, b, c] // dS[b] // dm[c, b, a]}
```

$$\left\{\begin{pmatrix} \omega & s_a & s_S \\ s_a & 1 & 0 \\ s_S & 0 & \Xi[1] \\ \Sigma & 1 & \sigma_S \end{pmatrix}, \begin{pmatrix} \frac{\omega\,\alpha[1]}{\sigma_a} & s_a & s_S \\ s_a & 1 & \frac{\theta[1]}{\alpha[1]} \\ s_S & 0 & \frac{\alpha[1]\,\Xi[1]-\theta[1]\,\phi[1]}{\alpha[1]} \\ \Sigma & 1 & \sigma_S \end{pmatrix}, \begin{pmatrix} \omega & s_a & s_S \\ s_a & 1 & \theta[1] \\ s_S & 0 & \Xi[1] \\ \Sigma & 1 & \sigma_S \end{pmatrix}, \begin{pmatrix} \omega & s_a & s_S \\ s_a & 1 & \theta[1] \\ s_S & 0 & \Xi[1] \\ \Sigma & 1 & \sigma_S \end{pmatrix}, \begin{pmatrix} \frac{\omega\,\alpha[1]}{\sigma_a} & s_a & s_S \\ s_a & 1 & \frac{\theta[1]}{\alpha[1]} \\ s_S & 0 & \frac{\alpha[1]\,\Xi[1]-\theta[1]\,\phi[1]}{\alpha[1]} \\ \Sigma & 1 & \sigma_S \end{pmatrix}\right\}$$

```
{Xp[S, a] // Γ, Xp[S, a] // Γ // qΔ[a, b, c] // dS[c] // dm[c, b, a]}
```

$$\left\{\begin{pmatrix} 1 & s_a & s_S \\ s_a & T_S & 0 \\ s_S & 1-T_S & 1 \\ \Sigma & T_S & 1 \end{pmatrix}, \begin{pmatrix} 1 & s_a & s_S \\ s_a & 1 & 0 \\ s_S & 0 & 1 \\ \Sigma & 1 & 1 \end{pmatrix}\right\}$$

```
Clear[α, θ, ϕ, Ξ, ω];
γ0 = Γ[ω, h_a σ_a + h_S σ_S, {t_a, t_S}.( α  θ ; ϕ  Ξ ).{h_a, h_S}];
{t1 = γ0 // qΔ[a, b, c] // dS[b] // dS[c],
 t2 = γ0 // dS[a] // qΔ[a, c, b], Simplify[t1 == t2]}
```

$$\left\{\begin{pmatrix} \frac{\alpha\,\omega}{\sigma_a} & s_b & s_c & s_S \\ s_b & \frac{-\alpha\,T_b+\alpha\,T_b\,T_c-\sigma_a+T_b\,\sigma_a}{\alpha\,(-1+T_b\,T_c)\,\sigma_a} & -\frac{(-1+T_b)\,(\alpha-\sigma_a)}{\alpha\,(-1+T_b\,T_c)\,\sigma_a} & \frac{\theta\,(-1+T_b)}{\alpha\,(-1+T_b\,T_c)} \\ s_c & -\frac{T_b\,(-1+T_c)\,(\alpha-\sigma_a)}{\alpha\,(-1+T_b\,T_c)\,\sigma_a} & \frac{-\alpha+\alpha\,T_b-T_b\,\sigma_a+T_b\,T_c\,\sigma_a}{\alpha\,(-1+T_b\,T_c)\,\sigma_a} & \frac{\theta\,T_b\,(-1+T_c)}{\alpha\,(-1+T_b\,T_c)} \\ s_S & -\frac{\phi}{\alpha} & -\frac{\phi}{\alpha} & \frac{\alpha\,\Xi-\theta\,\phi}{\alpha} \\ \Sigma & \frac{1}{\sigma_a} & \frac{1}{\sigma_a} & \sigma_S \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} \frac{\alpha\,\omega}{\sigma_a} & s_b & s_c & s_S \\ s_b & \frac{-\alpha\,T_b+\alpha\,T_b\,T_c-\sigma_a+T_b\,\sigma_a}{\alpha\,(-1+T_b\,T_c)\,\sigma_a} & -\frac{(-1+T_b)\,(\alpha-\sigma_a)}{\alpha\,(-1+T_b\,T_c)\,\sigma_a} & \frac{\theta\,(-1+T_b)}{\alpha\,(-1+T_b\,T_c)} \\ s_c & -\frac{T_b\,(-1+T_c)\,(\alpha-\sigma_a)}{\alpha\,(-1+T_b\,T_c)\,\sigma_a} & \frac{-\alpha+\alpha\,T_b-T_b\,\sigma_a+T_b\,T_c\,\sigma_a}{\alpha\,(-1+T_b\,T_c)\,\sigma_a} & \frac{\theta\,T_b\,(-1+T_c)}{\alpha\,(-1+T_b\,T_c)} \\ s_S & -\frac{\phi}{\alpha} & -\frac{\phi}{\alpha} & \frac{\alpha\,\Xi-\theta\,\phi}{\alpha} \\ \Sigma & \frac{1}{\sigma_a} & \frac{1}{\sigma_a} & \sigma_S \end{pmatrix}, \text{True}\right\}$$

dA tests for Γ

```
{Xp[1, 2] // Γ, (Xm[1, 2] // Γ) /. T₁ → 1 / T₁, Xm[1, 2] // Γ // dA[1] // dA[2]}
```

$$\left\{\begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & 1-T_1 \\ s_2 & 0 & T_1 \\ \Sigma & 1 & T_1 \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & 1-T_1 \\ s_2 & 0 & T_1 \\ \Sigma & 1 & T_1 \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & 1-T_1 \\ s_2 & 0 & T_1 \\ \Sigma & 1 & T_1 \end{pmatrix}\right\}$$

```
{Xm[1, 2] // Γ, (Xp[1, 2] // Γ) /. T₁ → 1 / T₁, Xp[1, 2] // Γ // dA[1] // dA[2]}
```

$$\left\{\begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & \frac{-1+T_1}{T_1} \\ s_2 & 0 & \frac{1}{T_1} \\ \Sigma & 1 & \frac{1}{T_1} \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & \frac{-1+T_1}{T_1} \\ s_2 & 0 & \frac{1}{T_1} \\ \Sigma & 1 & \frac{1}{T_1} \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & \frac{-1+T_1}{T_1} \\ s_2 & 0 & \frac{1}{T_1} \\ \Sigma & 1 & \frac{1}{T_1} \end{pmatrix}\right\}$$

```
Clear[α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ, μ, ω];
```

$$\left\{\gamma 0 = \Gamma\left[\omega,\ h_a\,\sigma_a + h_b\,\sigma_b + h_S\,\sigma_S,\ \{t_a, t_b, t_S\}.\begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \Xi \end{pmatrix}.\{h_a, h_b, h_S\}\right],\right.$$
$$\left. t1 = \gamma 0\ //\ dm[a, b, c]\ //\ dA[c],\ t2 = \gamma 0\ //\ dA[a]\ //\ dA[b]\ //\ dm[b, a, c],\ t1 == t2\right\}$$

$$\left\{\begin{pmatrix} \omega & s_a & s_b & s_S \\ s_a & \alpha & \beta & \theta \\ s_b & \gamma & \delta & \epsilon \\ s_S & \phi & \psi & \Xi \\ \Sigma & \sigma_a & \sigma_b & \sigma_S \end{pmatrix}, \begin{pmatrix} -\frac{(-\gamma+\beta\,\gamma-\alpha\,\delta)\,\omega}{\sigma_a\,\sigma_b} & s_c & s_S \\ s_c & \frac{-1+\beta}{-\gamma+\beta\,\gamma-\alpha\,\delta} & \frac{-\epsilon+\beta\,\epsilon-\delta\,\theta}{-\gamma+\beta\,\gamma-\alpha\,\delta} \\ s_S & -\frac{-\phi+\beta\,\phi-\alpha\,\psi}{-\gamma+\beta\,\gamma-\alpha\,\delta} & \frac{-\gamma\,\Xi+\beta\,\gamma\,\Xi-\alpha\,\delta\,\Xi+\epsilon\,\phi-\beta\,\epsilon\,\phi+\delta\,\theta\,\phi+\alpha\,\epsilon\,\psi-\gamma\,\theta\,\psi}{-\gamma+\beta\,\gamma-\alpha\,\delta} \\ \Sigma & \frac{1}{\sigma_a\,\sigma_b} & \sigma_S \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} -\frac{(-\gamma+\beta\,\gamma-\alpha\,\delta)\,\omega}{\sigma_a\,\sigma_b} & s_c & s_S \\ s_c & \frac{-1+\beta}{-\gamma+\beta\,\gamma-\alpha\,\delta} & \frac{-\epsilon+\beta\,\epsilon-\delta\,\theta}{-\gamma+\beta\,\gamma-\alpha\,\delta} \\ s_S & -\frac{-\phi+\beta\,\phi-\alpha\,\psi}{-\gamma+\beta\,\gamma-\alpha\,\delta} & \frac{-\gamma\,\Xi+\beta\,\gamma\,\Xi-\alpha\,\delta\,\Xi+\epsilon\,\phi-\beta\,\epsilon\,\phi+\delta\,\theta\,\phi+\alpha\,\epsilon\,\psi-\gamma\,\theta\,\psi}{-\gamma+\beta\,\gamma-\alpha\,\delta} \\ \Sigma & \frac{1}{\sigma_a\,\sigma_b} & \sigma_S \end{pmatrix}, \text{True}\right\}$$

```
Clear[α, θ, ϕ, Ξ, ω];
γ0 = Γ[ω, h_a σ_a + h_S σ_S, {t_a, t_S}.(α θ; ϕ Ξ).{h_a, h_S}];
{t1 = γ0 // qΔ[a, b, c] // dA[b] // dA[c],
 t2 = γ0 // dA[a] // qΔ[a, b, c], Simplify[t1 == t2]}
```

$$\left\{\begin{pmatrix} \frac{\alpha\,\omega}{\sigma_a} & S_b & S_c & S_S \\ S_b & \frac{-\alpha+\alpha\,T_c-T_c\,\sigma_a+T_b\,T_c\,\sigma_a}{\alpha\,(-1+T_b\,T_c)\,\sigma_a} & -\frac{(-1+T_b)\,T_c\,(\alpha-\sigma_a)}{\alpha\,(-1+T_b\,T_c)\,\sigma_a} & \frac{\theta\,(-1+T_b)\,T_c}{\alpha\,(-1+T_b\,T_c)} \\ S_c & -\frac{(-1+T_c)\,(\alpha-\sigma_a)}{\alpha\,(-1+T_b\,T_c)\,\sigma_a} & \frac{-\alpha\,T_c+\alpha\,T_b\,T_c-\sigma_a+T_c\,\sigma_a}{\alpha\,(-1+T_b\,T_c)\,\sigma_a} & \frac{\theta\,(-1+T_c)}{\alpha\,(-1+T_b\,T_c)} \\ S_S & -\frac{\phi}{\alpha} & -\frac{\phi}{\alpha} & \frac{\alpha\,\Xi-\theta\,\phi}{\alpha} \\ \Sigma & \frac{1}{\sigma_a} & \frac{1}{\sigma_a} & \sigma_S \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} \frac{\alpha\,\omega}{\sigma_a} & S_b & S_c & S_S \\ S_b & \frac{-\alpha+\alpha\,T_c-T_c\,\sigma_a+T_b\,T_c\,\sigma_a}{\alpha\,(-1+T_b\,T_c)\,\sigma_a} & -\frac{(-1+T_b)\,T_c\,(\alpha-\sigma_a)}{\alpha\,(-1+T_b\,T_c)\,\sigma_a} & \frac{\theta\,(-1+T_b)\,T_c}{\alpha\,(-1+T_b\,T_c)} \\ S_c & -\frac{(-1+T_c)\,(\alpha-\sigma_a)}{\alpha\,(-1+T_b\,T_c)\,\sigma_a} & \frac{-\alpha\,T_c+\alpha\,T_b\,T_c-\sigma_a+T_c\,\sigma_a}{\alpha\,(-1+T_b\,T_c)\,\sigma_a} & \frac{\theta\,(-1+T_c)}{\alpha\,(-1+T_b\,T_c)} \\ S_S & -\frac{\phi}{\alpha} & -\frac{\phi}{\alpha} & \frac{\alpha\,\Xi-\theta\,\phi}{\alpha} \\ \Sigma & \frac{1}{\sigma_a} & \frac{1}{\sigma_a} & \sigma_S \end{pmatrix}, \text{True}\right\}$$

```
n = 4; γ0 = Γ[ω, Sum[h_a σ_a, {a, 0, n}], Sum[t_a h_b α_(10 a+b), {a, 1, n}, {b, 1, n}]]
```

$$\begin{pmatrix} \omega & S_1 & S_2 & S_3 & S_4 \\ S_1 & \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ S_2 & \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ S_3 & \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\ S_4 & \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \\ \Sigma & \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 \end{pmatrix}$$

```
γ0 // dA[1] // dA[2] // dA[3] // dA[4]
```

$\omega\ (\alpha_{14}\,\alpha_{23}\,\alpha_{32}\,\alpha_{41}-\alpha_{13}\,\alpha_{24}\,\alpha_{32}\,\alpha_{41}-\alpha_{14}\,\alpha_{22}\,\alpha_{33}\,\alpha_{41}+\alpha_{12}\,\alpha_{24}\,\alpha_{33}\,\alpha_{41}+\alpha_{13}\,\alpha_{22}\,\alpha_{34}\,\alpha_{41}-\alpha_{12}\,\alpha_{23}\,\alpha_{34}\,\alpha_{41}-\alpha_{14}\,\alpha_{23}\,\alpha_{31}\,\alpha_{42}+\alpha_{13}\,\alpha_{24}\,\alpha_{31}\,\alpha_{42}+\alpha_{14}\,\alpha_{21}\,\alpha_{33}\,\alpha_{42}-\alpha_{11}\ (\ldots$

```
(γ0 // dA[1] // dA[2] // dA[3] // dA[4]) ** γ0
```

$\omega^2\ (\alpha_{14}\,\alpha_{23}\,\alpha_{32}\,\alpha_{41}-\alpha_{13}\,\alpha_{24}\,\alpha_{32}\,\alpha_{41}-\alpha_{14}\,\alpha_{22}\,\alpha_{33}\,\alpha_{41}+\alpha_{12}\,\alpha_{24}\,\alpha_{33}\,\alpha_{41}+\alpha_{13}\,\alpha_{22}\,\alpha_{34}\,\alpha_{41}-\alpha_{12}\,\alpha_{23}\,\alpha_{34}\,\alpha_{41}-\alpha_{14}\,\alpha_{23}\,\alpha_{31}\,\alpha_{42}+\alpha_{13}\,\alpha_{24}\,\alpha_{31}\,\alpha_{42}+\alpha_{14}\,\alpha_{21}\,\alpha_{33}\,\alpha_{42}-\alpha_{11}\ \ldots$

$$(\alpha_{14}\,\alpha_{23}\,\alpha_{32}\,\alpha_{41} - \alpha_{13}\,\alpha_{24}\,\alpha_{32}\,\alpha_{41} - \alpha_{14}\,\alpha_{22}\,\alpha_{33}\,\alpha_{41} + \alpha_{12}\,\alpha_{24}\,\alpha_{33}\,\alpha_{41} + \alpha_{13}\,\alpha_{22}\,\alpha_{34}\,\alpha_{41} - \alpha_{12}\,\alpha_{23}\,\alpha_{34}\,\alpha_{41} - \alpha_{14}\,\alpha_{23}\,\alpha_{31}\,\alpha_{42} + \alpha_{13}\,\alpha_{24}\,\alpha_{31}\,\alpha_{42} + \alpha_{14}\,\alpha_{21}\,\alpha_{33}\,\alpha_{42} - \alpha_{11}\,\alpha_{24}\,\alpha_{33}\,\alpha_{42} - \alpha_{13}\,\alpha_{21}\,\alpha_{34}\,\alpha_{42} + \alpha_{11}\,\alpha_{23}\,\alpha_{34}\,\alpha_{42} + \alpha_{14}\,\alpha_{22}\,\alpha_{31}\,\alpha_{43} - \alpha_{12}\,\alpha_{24}\,\alpha_{31}\,\alpha_{43} - \alpha_{14}\,\alpha_{21}\,\alpha_{32}\,\alpha_{43} + \alpha_{11}\,\alpha_{24}\,\alpha_{32}\,\alpha_{43} + \alpha_{12}\,\alpha_{21}\,\alpha_{34}\,\alpha_{43} - \alpha_{11}\,\alpha_{22}\,\alpha_{34}\,\alpha_{43} - \alpha_{13}\,\alpha_{22}\,\alpha_{31}\,\alpha_{44} + \alpha_{12}\,\alpha_{23}\,\alpha_{31}\,\alpha_{44} + \alpha_{13}\,\alpha_{21}\,\alpha_{32}\,\alpha_{44} - \alpha_{11}\,\alpha_{23}\,\alpha_{32}\,\alpha_{44} - \alpha_{12}\,\alpha_{21}\,\alpha_{33}\,\alpha_{44} + \alpha_{11}\,\alpha_{22}\,\alpha_{33}\,\alpha_{44}) == \mathrm{Det}\left[\begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \end{pmatrix}\right]$$

```
True
```

Θ tests

```
Θ[1, 2] // A
```

$$\begin{pmatrix} 1 & h[1] & h[2] \\ t[1] & \frac{-c_1 + e^{\frac{c_2}{2}}\,c_1 - e^{\frac{c_1}{2}+\frac{c_2}{2}}\,c_2 + e^{\frac{c_2}{2}}\,c_2}{c_1^2 + c_1\,c_2} & \frac{-1 + e^{\frac{c_1}{2}+\frac{c_2}{2}}}{c_1 + c_2} \\ t[2] & \frac{-1 + e^{\frac{c_1}{2}+\frac{c_2}{2}}}{c_1 + c_2} & \frac{e^{\frac{c_1}{2}}\,c_1 - e^{\frac{c_1}{2}+\frac{c_2}{2}}\,c_1 - c_2 + e^{\frac{c_1}{2}}\,c_2}{c_1\,c_2 + c_2^2} \end{pmatrix}$$

```
(V // A) ** (Θ[1, 2] // A)
```

$$\begin{pmatrix} \frac{2^{1/4}\left(\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}\right)^{1/4}\left(\frac{\mathrm{Sinh}\left[\frac{c_2}{2}\right]}{c_2}\right)^{1/4}}{\left(\frac{\mathrm{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}\right)^{1/4}} & h[1] \\ t[1] & \frac{-\sqrt{2}\,e^{\frac{c_1}{2}+\frac{c_2}{2}}\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_2}{2}\right]}{c_2}}\,c_2 - 2\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_1\sqrt{\frac{\mathrm{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}} + 2\,e^{\frac{c_2}{2}}\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_1\sqrt{\frac{\mathrm{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}} + 2\,e^{\frac{c_2}{2}}\cdots}{2\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_1^2\sqrt{\frac{\mathrm{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}} + 2\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_1\,c_2\sqrt{\frac{\mathrm{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}} \\ t[2] & \frac{\sqrt{2}\,e^{\frac{c_1}{2}+\frac{c_2}{2}}\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_2}{2}\right]}{c_2}} - 2\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\sqrt{\frac{\mathrm{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}}{2\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_1\sqrt{\frac{\mathrm{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}} + 2\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_2\sqrt{\frac{\mathrm{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}} \end{pmatrix}$$

```
(Xp[1, 2] // A) ** (V // A // dσ[1 → 2, 2 → 1])
```

$$\left(\begin{array}{ll} \dfrac{2^{1/4}\left(\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}\right)^{1/4}\left(\frac{\text{Sinh}\left[\frac{c_2}{2}\right]}{c_2}\right)^{1/4}}{\left(\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}\right)^{1/4}} & h[1] \\ t[1] & \dfrac{\sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\, c_1 - e^{\frac{c_2}{2}}\sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\, c_1 - e^{c_1+c_2}\sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\, c_1 + e^{c_1+\frac{3c_2}{2}}\sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\, c_1 - e^{\frac{c_2}{2}}\sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\, c_2 + e^{c_1+\frac{3c_2}{2}}\sqrt{\cdots}}{-\sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\, c_1^2 + e^{c_1+c_2}\sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\, c_1^2 - \sqrt{\cdots}} \\ t[2] & \dfrac{\sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}} - e^{c_1+c_2}\sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}} + \sqrt{2}\, e^{c_1+c_2} c_1 \sqrt{\frac{\text{Sinh}\left[\frac{c_2}{2}\right]}{c_2}}\sqrt{\cdots}}{-\sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\, c_1 + e^{c_1+c_2}\sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\, c_1 - \sqrt{\cdots}} \end{array}\right.$$

```
(V // A) ** (Θ[1, 2] // A) == (Xp[1, 2] // A) ** (V // A // dσ[1 → 2, 2 → 1]) // Simplify
```

True

```
{t1 = Θ[1, 2] // A // Γ, t2 = Θi[1, 2] // A // Γ, t1 ** t2}
```

$$\left\{\begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & \frac{\text{Log}[T_1]+\text{Log}[T_2]\sqrt{T_1}\sqrt{T_2}}{\text{Log}[T_1]+\text{Log}[T_2]} & -\frac{\text{Log}[T_1]\left(-1+\sqrt{T_1}\sqrt{T_2}\right)}{\text{Log}[T_1]+\text{Log}[T_2]} \\ s_2 & -\frac{\text{Log}[T_2]\left(-1+\sqrt{T_1}\sqrt{T_2}\right)}{\text{Log}[T_1]+\text{Log}[T_2]} & \frac{\text{Log}[T_2]+\text{Log}[T_1]\sqrt{T_1}\sqrt{T_2}}{\text{Log}[T_1]+\text{Log}[T_2]} \\ \Sigma & \sqrt{T_2} & \sqrt{T_1} \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & \frac{\text{Log}[T_2]+\text{Log}[T_1]\sqrt{T_1}\sqrt{T_2}}{(\text{Log}[T_1]+\text{Log}[T_2])\sqrt{T_1}\sqrt{T_2}} & \frac{\text{Log}[T_1]\left(-1+\sqrt{T_1}\sqrt{T_2}\right)}{(\text{Log}[T_1]+\text{Log}[T_2])\sqrt{T_1}\sqrt{T_2}} \\ s_2 & \frac{\text{Log}[T_2]\left(-1+\sqrt{T_1}\sqrt{T_2}\right)}{(\text{Log}[T_1]+\text{Log}[T_2])\sqrt{T_1}\sqrt{T_2}} & \frac{\text{Log}[T_1]+\text{Log}[T_2]\sqrt{T_1}\sqrt{T_2}}{(\text{Log}[T_1]+\text{Log}[T_2])\sqrt{T_1}\sqrt{T_2}} \\ \Sigma & \frac{1}{\sqrt{T_2}} & \frac{1}{\sqrt{T_1}} \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & 0 \\ s_2 & 0 & 1 \\ \Sigma & 1 & 1 \end{pmatrix}\right\}$$

```
(V // A) ** (Θi[1, 2] // A) ==
  (Xm[2, 1] // A) ** (V // A // dσ[1 → 2, 2 → 1]) // FullSimplify
```

True

```
(Θ[1, 2] // A) == (Vi // A) ** (Xp[1, 2] // A) ** (V // A // dσ[1 → 2, 2 → 1]) // Simplify
```

Simplify::time :
Time spent on a transformation exceeded 300.` seconds, and the transformation was aborted. Increasing the value of TimeConstraint option may improve the result of simplification. ≫

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Time spent on a transformation exceeded 300.` seconds, and the transformation was aborted. Increasing the value of TimeConstraint option may improve the result of simplification. ≫

Simplify::time :
Time spent on a transformation exceeded 300.` seconds, and the transformation was aborted. Increasing the value of TimeConstraint option may improve the result of simplification. ≫

General::stop : Further output of Simplify::time will be suppressed during this calculation. ≫

```
$Aborted
```

```
(Xp[2, 1] // A) == (V // A // dσ[1 → 2, 2 → 1]) ** (Θ[1, 2] // A) ** (Vi // A) // Simplify
```

```
True
```

Γb-Calculus

```
Clear[α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ, μ, ω];
```

$$\left\{\gamma 0 = \Gamma b\left[\omega,\ h_a\,\sigma_a + h_b\,\sigma_b + h_S\,\sigma_S,\ \{t_a, t_b, t_S\}.\begin{pmatrix}\alpha & \beta & \theta\\ \gamma & \delta & \epsilon\\ \phi & \psi & \Xi\end{pmatrix}.\{h_a, h_b, h_S\}\right],\ \gamma 0 \mathbin{//} \Gamma \mathbin{//} dm[a, b, c] \mathbin{//} \Gamma b\right\}$$

$$\left\{\begin{pmatrix}\omega & s_a & s_b & s_S\\ s_a & \alpha & \beta & \theta\\ s_b & \gamma & \delta & \epsilon\\ s_S & \phi & \psi & \Xi\\ \Sigma & \sigma_a & \sigma_b & \sigma_S\end{pmatrix},\ \begin{pmatrix}-\beta+\omega & s_c & s_S\\ s_c & \frac{-\beta\,\gamma+\alpha\,\delta+\gamma\,\omega}{\omega} & \frac{-\beta\,\epsilon+\delta\,\theta+\epsilon\,\omega}{\omega}\\ s_S & \frac{-\beta\,\phi+\alpha\,\psi+\phi\,\omega}{\omega} & \frac{-\beta\,\Xi+\theta\,\psi+\Xi\,\omega}{\omega}\\ \Sigma & \sigma_a\,\sigma_b & \sigma_S\end{pmatrix}\right\}$$

```
V // A // Γb // ΓbCollect[FullSimplify[PowerExpand[#]] &]
```

$$\left(\begin{matrix}\frac{(\mathrm{Log}[T_1]+\mathrm{Log}[T_2])^{1/4}\,(-1+T_1)^{1/4}\,(-1+T_2)^{1/4}}{\mathrm{Log}[T_1]^{1/4}\,\mathrm{Log}[T_2]^{1/4}\,(-1+T_1\,T_2)^{1/4}} & \\ s_1 & \frac{\mathrm{Log}[T_1]^{1/4}\left(\sqrt{\mathrm{Log}[T_1]}\,\sqrt{\mathrm{Log}[T_1]+\mathrm{Log}[T_2]}\,\sqrt{-1+T_2}-\sqrt{\mathrm{Log}[T_1]}\,\sqrt{\mathrm{Log}[T_1]+\mathrm{Log}[T_2]}\,T_1\right.}{}\\ s_2 & \\ \Sigma & \end{matrix}\right.$$

```
V // A // Γb // ΓbCollect[FullSimplify[PowerExpand[#]] &] // ΓbCollect[
  Assuming[c1 > 0 && c2 > 0, FullSimplify[# /. Log[x_] :> Log[x /. Ta_ :> ⅇ^ca]]] &]
```

$$\begin{pmatrix} \frac{((c_1+c_2)(-1+T_1))^{1/4}(-1+T_2)^{1/4}}{(c_1 c_2(-1+T_1 T_2))^{1/4}} & & \\ s_1 & & \frac{\sqrt{c_1(c_1+c_2)(-1+T_2)} - T_1\sqrt{c_1(c_1+c_2)(-1+T_2)} - T_1\sqrt{c_1(c_1+c_2)(-1+T_2)}\,T_2 + T_1^2\sqrt{c_1(c_1+c_2)(-1+T_2)}}{(-1+} \\ s_2 & & \frac{c_2^{1/4}(-1+T_2)^{1/4}}{} \\ \Sigma & & \end{pmatrix}$$

```
V // A // Γ // ΓCollect[FullSimplify[PowerExpand[#]] &] // ΓCollect[
  Assuming[c1 > 0 && c2 > 0, FullSimplify[# /. Log[x_] :> Log[x /. Ta_ :> ⅇ^ca]]] &]
```

$$\begin{pmatrix} \frac{((c_1+c_2)(-1+T_1))^{1/4}(-1+T_2)^{1/4}}{(c_1 c_2(-1+T_1 T_2))^{1/4}} & s_1 & \\ s_1 & \frac{c_1\sqrt{c_1 c_2(-1+T_2)} + c_2\sqrt{c_1 c_2(-1+T_2)} + c_1\sqrt{\frac{(c_1+c_2)(-1+T_1)}{T_1}}\sqrt{-1+T_1 T_2}}{(c_1+c_2)\sqrt{\frac{(c_1+c_2)(-1+T_1)}{T_1}}\sqrt{-1+T_1 T_2}} & \frac{-\sqrt{c_1 c_2(c_1+c_2)(-1+T_1)}}{(c_1+c_2)\sqrt{-}} \\ s_2 & \frac{-c_1\sqrt{c_1 c_2(-1+T_2)} - c_2\sqrt{c_1 c_2(-1+T_2)} + c_2\sqrt{\frac{(c_1+c_2)(-1+T_1)}{T_1}}\sqrt{-1+T_1 T_2}}{(c_1+c_2)\sqrt{\frac{(c_1+c_2)(-1+T_1)}{T_1}}\sqrt{-1+T_1 T_2}} & \frac{\sqrt{c_1 c_2(c_1+c_2)(-1+T_1)}}{(c_1+c_2)\sqrt{-}} \\ \Sigma & 1 & \end{pmatrix}$$

```
V // A // Γ // ΓCollect[FullSimplify[PowerExpand[#]] &] // ΓCollect[
  Assuming[d1 > 0 && d2 > 0, FullSimplify[# /. Log[x_] :> Log[x /. Ta_ :> ⅇ^(da^2)]]] &]
```

$$\begin{pmatrix} \frac{((d_1^2+d_2^2)(-1+T_1))^{1/4}(-1+T_2)^{1/4}}{(d_1^2 d_2^2(-1+T_1 T_2))^{1/4}} & s_1 & s \\ s_1 & \frac{d_1\left(d_1^2 d_2\sqrt{-1+T_2} + d_2^3\sqrt{-1+T_2} + d_1\sqrt{\frac{-1+T_1}{T_1}}\sqrt{(d_1^2+d_2^2)(-1+T_1 T_2)}\right)}{(d_1^2+d_2^2)\sqrt{\frac{-1+T_1}{T_1}}\sqrt{(d_1^2+d_2^2)(-1+T_1 T_2)}} & \frac{d_1\left(-d_2\sqrt{(d_1^2+d_2^2)(-1+T_1)}\sqrt{T}\right.}{(d_1^2+d_2^2)\sqrt{-1+}} \\ s_2 & -\frac{d_2\left(d_1^3\sqrt{-1+T_2} + d_1 d_2^2\sqrt{-1+T_2} - d_2\sqrt{\frac{-1+T_1}{T_1}}\sqrt{(d_1^2+d_2^2)(-1+T_1 T_2)}\right)}{(d_1^2+d_2^2)\sqrt{\frac{-1+T_1}{T_1}}\sqrt{(d_1^2+d_2^2)(-1+T_1 T_2)}} & \frac{d_2\left(d_1\sqrt{(d_1^2+d_2^2)(-1+T_1)}\sqrt{T}\right.}{(d_1^2+d_2^2)\sqrt{-1+}} \\ \Sigma & 1 & \sqrt{-} \end{pmatrix}$$