Oberwolfach Handout on May 15, 2014

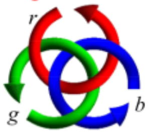
May-15-14 4:04 PM

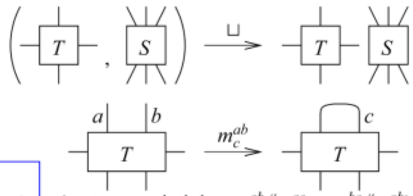
Some very good formulas for the Alexander polynomial, 1

Abstract. I will describe some very good formulas for a *(matrix plus scalar)*-valued extension of the Alexander polynomnial to tangles, then say that everything extends to virtual tangles, then roughly to simply knotted balloons and hoops in 4D, then the target space extends to *(free Lie algebras plus cyclic words)*, and the result is a universal finite type of the knotted objects in its domain. Taking a cue from the BF topological quantum field theory, everything should extend (with some modifications) to arbitrary codimension-2 knots in arbitrary dimension and in particular, to arbitrary 2-knots in 4D. But what is really going on is still a mystery.

runs

Tangles.





(meta-associativity: $m^{ab}_a /\!/ m^{ac}_a = m^{bc}_b /\!/ m^{ab}_a$)

KBH

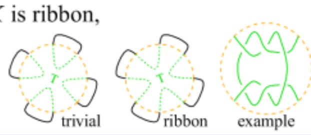
Pictures from Geneva talk

Why Tangles?

- Finitely presented.
- Divide and conquer proofs and computations.
- "Algebraic Knot Theory": If K is ribbon,

$Z(K) \in \{cl_2(Z) : cl_1(Z) = 1\}$.

(Genus and crossing number are also definable properties).



2 →

Theorem 1. $\exists!$ an invariant γ: {pure S-tangles} $\to R \times M_{S\times S}(R)$, where $R = R_S = \mathbb{Z}((T_a)_{a\in S})$ is the ring of rational functions in S variables, intertwining

$$\textbf{1.}\ \left(\begin{array}{c|c}\omega_1 & S_1\\ \hline S_1 & A_1\end{array},\ \begin{array}{c|c}\omega_2 & S_2\\ \hline S_2 & A_2\end{array}\right) \xrightarrow{\sqcup} \begin{array}{c|cc}\omega_1\omega_2 & S_1 & S_2\\ \hline S_1 & A_1 & 0\\ S_2 & 0 & A_2\end{array},$$

$$\textbf{2.}\ \begin{array}{c|ccc}\omega & a & b & S\\ \hline a & \alpha & \beta & \theta\\ b & \gamma & \delta & \epsilon\\ S & \phi & \psi & \Xi\end{array} \xrightarrow[\mu:=1-\beta]{m^{ab}_c} \left(\begin{array}{c|cc}\mu\omega & c & S\\ \hline c & \gamma+\alpha\delta/\mu & \epsilon+\delta\theta/\mu\\ S & \phi+\alpha\psi/\mu & \Xi+\theta\psi/\mu\end{array}\right)_{T_a,T_b\to T_c},$$

and satisfying $\left(\uparrow_a;\ {}_a\!\nearrow\!\!\!\nwarrow_b,\ {}_b\!\nearrow\!\!\!\nwarrow_a\right) \xrightarrow{\gamma} \left(\begin{array}{c|c}1 & a\\ \hline a & 1\end{array};\ \begin{array}{c|cc}1 & a & b\\ \hline a & 1 & 1-T_a^{\pm1}\\ b & 0 & T_a^{\pm1}\end{array}\right)$.

β formulas from cheat sheet

In Addition, • This is really "just" a stitching formula for Burau/Gassner. / 3 modern refs

- $L \mapsto (\omega, A) \mapsto \omega\det'(A-I)/(1-T)$ is the MVA, mod units.
- The "fastest" Alexander algorithm.
- There are also formulas for strand deletion, reversal, and doubling.
- Every step along the computation is the invariant of something.
- Fits in one column, including propaganda & implementation:

show 2₀

β ↔ Gassner

```
ΓSimp = Factor; SetAttributes[ΓCollect, Listable];
ΓCollect[Γ[ω_, λ_]] := Γ[ΓSimp[ω],
   Collect[λ, h_, Collect[#, t_, ΓSimp] &]];
ΓForm[Γ[ω_, λ_]] := Module[{S, M},
   S = Union[Cases[B[ω, λ], (h | t)a_ :> a, Infinity]];
   M = Outer[ΓSimp[Coefficient[λ, hα1 tα2]] &, S, S];
   PrependTo[M, sα & /@ S];
   M = Prepend[Transpose[M], Prepend[sα & /@ S, ω]];
   MatrixForm[M]];
ΓForm[else_] := else /. γ_Γ :> ΓForm[γ];
Format[γ_Γ, StandardForm] := ΓForm[γ];
```

```
Γ /: Γ[ω1_, λ1_] Γ[ω2_, λ2_] := Γ[ω1*ω2, λ1+λ2];
ma_,b_→c_[Γ[ω_, λ_]] := Module[{α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ, μ},
   (α β θ; γ δ ϵ; ϕ ψ Ξ) = (∂ta,ha λ  ∂ta,hb λ  ∂ta λ; ∂tb,ha λ  ∂tb,hb λ  ∂tb λ; ∂ha λ  ∂hb λ  λ) /. (t | h)a|b → 0;
   Γ[(μ = 1 - β) ω, {tc, 1}.(γ+αδ/μ  ϵ+δθ/μ; ϕ+αψ/μ  Ξ+θψ/μ).{hc, 1}]
     /. {Ta → Tc, Tb → Tc} // ΓCollect];
Rpa_,b_ := Γ[1, {ta, tb}.(1 1-Ta; 0 Ta).{ha, hb}];
Rma_,b_ := Rpab /. Ta → 1/Ta;
```

extends to more naturally defined on v/w-tangles.

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References.

[BN] D. Bar-Natan, *Balloons and Hoops and their Universal Finite Type Invariant, BF Theory, and an Ultimate Alexander Invariant,* http://www.math.toronto.edu/~drorbn/papers/KBH/, arXiv:1308.1721.

[BND] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects: From Alexander to Kashiwara and Vergne,* http://www.math.toronto.edu/~drorbn/papers/WKO/.

[CR] A. S. Cattaneo and C. A. Rossi, *Wilson Surfaces and Higher Dimensional Knot Invariants,* Commun. in Math. Phys. **256-3** (2005) 513–537, arXiv:math-ph/0210037.

split

+ Le Dimet, KLZ, CT

"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

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