

Pensieve header: Demos for my Oberwolfach-1405 talk.

The .nb (source) file is at <http://drorbn.net/AcademicPensieve/2014-05/GoodFormulas/>.

```
SetDirectory["C:/drorbn/AcademicPensieve/2014-05/GoodFormulas/"]
```

```
C:\drorbn\AcademicPensieve\2014-05\GoodFormulas
```

```
<< KnotTheory`
```

```
Loading KnotTheory` version of April 3, 2014, 16:23:56.0784.
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```
Read more at http://katlas.org/wiki/KnotTheory.
```

Initialization

```
ΓCollect[Γ[ω_, λ_]] := Γ[Factor[ω],
  Collect[λ, h_, Collect[#, t_, Factor] &]];
Format[Γ[ω_, λ_]] := Module[{S, M},
  S = Union@Cases[Γ[ω, λ], (h | t)_a_ -> a, ∞];
  M = Outer[Factor[∂_{h_i t_{i+2}} λ] &, S, S];
  M = Prepend[M, t_# & /@ S] // Transpose;
  M = Prepend[M, Prepend[h_# & /@ S, ω]];
  M // MatrixForm];
```

Program

```
Γ /: Γ[ω1_, λ1_] Γ[ω2_, λ2_] := Γ[ω1 * ω2, λ1 + λ2];
```

```
m_{a_b -> c}_[Γ[ω_, λ_]] := Module[{α, β, γ, δ, θ, ε, φ, ψ, Ξ, μ},
```

$$\begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \Xi \end{pmatrix} = \begin{pmatrix} \partial_{t_a, h_a} \lambda & \partial_{t_a, h_b} \lambda & \partial_{t_a} \lambda \\ \partial_{t_b, h_a} \lambda & \partial_{t_b, h_b} \lambda & \partial_{t_b} \lambda \\ \partial_{h_a} \lambda & \partial_{h_b} \lambda & \lambda \end{pmatrix} / . (t | h)_{a|b} \rightarrow 0;$$

$$\Gamma[(\mu = 1 - \beta) \omega, \{t_c, 1\}] \cdot \begin{pmatrix} \gamma + \alpha \delta / \mu & \epsilon + \delta \theta / \mu \\ \phi + \alpha \psi / \mu & \Xi + \psi \theta / \mu \end{pmatrix} \cdot \{h_c, 1\} / . \{T_a \rightarrow T_c, T_b \rightarrow T_c\} // \GammaCollect];$$

```
Rp_{a_b} := Γ[1, {t_a, t_b}] \cdot \begin{pmatrix} 1 & 1 - T_a \\ 0 & T_a \end{pmatrix} \cdot \{h_a, h_b\};
```

```
Rm_{a_b} := Rp_{ab} / . T_a \rightarrow 1 / T_a;
```

$$\gamma = \Gamma[\omega, (t_1 \ t_2 \ t_3 \ t_s)] \cdot \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \theta_1 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \theta_2 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \theta_3 \\ \phi_1 & \phi_2 & \phi_3 & \Xi \end{pmatrix} \cdot \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_s \end{pmatrix} // \text{Total} // \text{Total}$$

$$\begin{pmatrix} \omega & h_1 & h_2 & h_3 & h_s \\ t_1 & \alpha_{11} & \alpha_{12} & \alpha_{13} & \theta_1 \\ t_2 & \alpha_{21} & \alpha_{22} & \alpha_{23} & \theta_2 \\ t_3 & \alpha_{31} & \alpha_{32} & \alpha_{33} & \theta_3 \\ t_s & \phi_1 & \phi_2 & \phi_3 & \Xi \end{pmatrix}$$

```
γ // m_{12 -> 1} // m_{13 -> 1}
```

$$\begin{pmatrix} \omega (1 - \alpha_{12} - \alpha_{13} \alpha_{22} - \alpha_{23} + \alpha_{12} \alpha_{23}) & h_1 \\ t_1 & \frac{\alpha_{31} - \alpha_{12} \alpha_{31} - \alpha_{13} \alpha_{22} \alpha_{31} - \alpha_{23} \alpha_{31} + \alpha_{12} \alpha_{23} \alpha_{31} + \alpha_{11} \alpha_{32} + \alpha_{13} \alpha_{21} \alpha_{32} - \alpha_{11} \alpha_{23} \alpha_{32} + \alpha_{21} \alpha_{33} - \alpha_{12} \alpha_{21} \alpha_{33} + 1 - \alpha_{12} - \alpha_{13} \alpha_{22} - \alpha_{23} + \alpha_{12} \alpha_{23}}{1 - \alpha_{12} - \alpha_{13} \alpha_{22} - \alpha_{23} + \alpha_{12} \alpha_{23}} \\ t_s & \frac{\phi_1 - \alpha_{12} \phi_1 - \alpha_{13} \alpha_{22} \phi_1 - \alpha_{23} \phi_1 + \alpha_{12} \alpha_{23} \phi_1 + \alpha_{11} \phi_2 + \alpha_{13} \alpha_{21} \phi_2 - \alpha_{11} \alpha_{23} \phi_2 + \alpha_{21} \phi_3 - \alpha_{12} \alpha_{21} \phi_3 + \alpha_{11}}{1 - \alpha_{12} - \alpha_{13} \alpha_{22} - \alpha_{23} + \alpha_{12} \alpha_{23}} \end{pmatrix}$$

$$\gamma // m_{23 \rightarrow 2} // m_{12 \rightarrow 1}$$

$$\left( \begin{array}{c} \omega (1 - \alpha_{12} - \alpha_{13} \alpha_{22} - \alpha_{23} + \alpha_{12} \alpha_{23}) \\ t_1 \\ t_s \end{array} \right) \begin{array}{c} h_1 \\ \frac{\alpha_{31} - \alpha_{12} \alpha_{31} - \alpha_{13} \alpha_{22} \alpha_{31} - \alpha_{23} \alpha_{31} + \alpha_{12} \alpha_{23} \alpha_{31} + \alpha_{11} \alpha_{32} + \alpha_{13} \alpha_{21} \alpha_{32} - \alpha_{11} \alpha_{23} \alpha_{32} + \alpha_{21} \alpha_{33} - \alpha_{12} \alpha_{21} \alpha_{33} +}{1 - \alpha_{12} - \alpha_{13} \alpha_{22} - \alpha_{23} + \alpha_{12} \alpha_{23}} \\ \frac{\phi_1 - \alpha_{12} \phi_1 - \alpha_{13} \alpha_{22} \phi_1 - \alpha_{23} \phi_1 + \alpha_{12} \alpha_{23} \phi_1 + \alpha_{11} \phi_2 + \alpha_{13} \alpha_{21} \phi_2 - \alpha_{11} \alpha_{23} \phi_2 + \alpha_{21} \phi_3 - \alpha_{12} \alpha_{21} \phi_3 + \alpha_{11}}{1 - \alpha_{12} - \alpha_{13} \alpha_{22} - \alpha_{23} + \alpha_{12} \alpha_{23}} \end{array}$$

MetaAssoc

$$\gamma = \Gamma \left[ \omega, \{t_1, t_2, t_3, t_s\} \cdot \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \theta_1 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \theta_2 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \theta_3 \\ \phi_1 & \phi_2 & \phi_3 & \Xi \end{pmatrix} \cdot \{h_1, h_2, h_3, h_s\} \right];$$

$$(\gamma // m_{12 \rightarrow 1} // m_{13 \rightarrow 1}) == (\gamma // m_{23 \rightarrow 2} // m_{12 \rightarrow 1})$$

MetaAssoc

True

R3

$$\{ \text{Rm}_{51} \text{Rm}_{62} \text{Rp}_{34} // m_{14 \rightarrow 1} // m_{25 \rightarrow 2} // m_{36 \rightarrow 3}, \\ \text{Rp}_{61} \text{Rm}_{24} \text{Rm}_{35} // m_{14 \rightarrow 1} // m_{25 \rightarrow 2} // m_{36 \rightarrow 3} \}$$

R3

$$\left\{ \begin{pmatrix} 1 & h_1 & h_2 & h_3 \\ t_1 & \frac{T_3}{T_2} & 0 & 0 \\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0 \\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1 \end{pmatrix}, \begin{pmatrix} 1 & h_1 & h_2 & h_3 \\ t_1 & \frac{T_3}{T_2} & 0 & 0 \\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0 \\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1 \end{pmatrix} \right\}$$

$$\gamma = \text{Rm}_{12,1} \text{Rm}_{27} \text{Rm}_{83} \text{Rm}_{4,11} \text{Rp}_{16,5} \text{Rp}_{6,13} \text{Rp}_{14,9} \text{Rp}_{10,15}$$

$$\left( \begin{array}{cccccccccccccccc} 1 & h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 & h_8 & h_9 & h_{10} & h_{11} & h_{12} & h_{13} & h_{14} & h_{15} & h_{16} \\ t_1 & \frac{1}{T_{12}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ t_2 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{-1+T_2}{T_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ t_3 & 0 & 0 & \frac{1}{T_8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ t_4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1+T_4}{T_4} & 0 & 0 & 0 & 0 & 0 \\ t_5 & 0 & 0 & 0 & 0 & T_{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ t_6 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 - T_6 & 0 & 0 & 0 \\ t_7 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{T_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ t_8 & 0 & 0 & \frac{-1+T_8}{T_8} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ t_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & T_{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ t_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 - T_{10} & 0 \\ t_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{T_4} & 0 & 0 & 0 & 0 & 0 \\ t_{12} & \frac{-1+T_{12}}{T_{12}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ t_{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & T_6 & 0 & 0 & 0 \\ t_{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 - T_{14} & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ t_{15} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & T_{10} & 0 \\ t_{16} & 0 & 0 & 0 & 0 & 1 - T_{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right)$$

Do[ $\gamma = \gamma // m_{1k \rightarrow 1}, \{k, 2, 10\}\}; \gamma$

$$\begin{pmatrix} \frac{T_1^2+T_{16}-T_1 T_{16}}{T_1^2} & h_1 & h_{11} & h_{12} & h_{13} & h_{14} & h_{15} & h_{16} \\ t_1 & \frac{T_{14} (-T_1+T_1^2+T_{16})}{T_{12} (T_1^2+T_{16}-T_1 T_{16})} & \frac{(-1+T_1) (1-T_1+T_1^2) T_{14} T_{16}}{T_1 (T_1^2+T_{16}-T_1 T_{16})} & 0 & -\frac{(-1+T_1) (1-T_1+T_1^2) T_{14}}{T_1^2+T_{16}-T_1 T_{16}} & 0 & 1-T_1 & 0 \\ t_{11} & 0 & \frac{1}{T_1} & 0 & 0 & 0 & 0 & 0 \\ t_{12} & \frac{-1+T_{12}}{T_{12}} & 0 & 1 & 0 & 0 & 0 & 0 \\ t_{13} & 0 & 0 & 0 & T_1 & 0 & 0 & 0 \\ t_{14} & -\frac{(-1+T_{14}) (-T_1+T_1^2+T_{16})}{T_{12} (T_1^2+T_{16}-T_1 T_{16})} & -\frac{(-1+T_1) (1-T_1+T_1^2) (-1+T_{14}) T_{16}}{T_1 (T_1^2+T_{16}-T_1 T_{16})} & 0 & \frac{(-1+T_1) (1-T_1+T_1^2) (-1+T_{14})}{T_1^2+T_{16}-T_1 T_{16}} & 1 & 0 & 0 \\ t_{15} & 0 & 0 & 0 & 0 & 0 & T_1 & 0 \\ t_{16} & -\frac{T_1 (-1+T_{16})}{T_{12} (T_1^2+T_{16}-T_1 T_{16})} & -\frac{(-1+T_1) T_1 (-1+T_{16})}{T_1^2+T_{16}-T_1 T_{16}} & 0 & \frac{(-1+T_1)^2 (-1+T_{16})}{T_1^2+T_{16}-T_1 T_{16}} & 0 & 0 & 1 \end{pmatrix}$$

8\_17

$\gamma = \text{Rm}_{12,1} \text{Rm}_{27} \text{Rm}_{83} \text{Rm}_{4,11} \text{Rp}_{16,5} \text{Rp}_{6,13} \text{Rp}_{14,9} \text{Rp}_{10,15};$

Do[ $\gamma = \gamma // m_{1k \rightarrow 1}, \{k, 2, 16\}\}; \gamma$

8\_17

$$\begin{pmatrix} -\frac{1-4 T_1+8 T_1^2-11 T_1^3+8 T_1^4-4 T_1^5+T_1^6}{T_1^3} & h_1 \\ t_1 & 1 \end{pmatrix}$$

$\gamma = \text{Rm}_{16} \text{Rp}_{74} \text{Rp}_{52} \text{Rm}_{8,11} \text{Rp}_{12,9} \text{Rp}_{10,3}$

$$\begin{pmatrix} 1 & h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 & h_8 & h_9 & h_{10} & h_{11} & h_{12} \\ t_1 & 1 & 0 & 0 & 0 & 0 & \frac{-1+T_1}{T_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ t_2 & 0 & T_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ t_3 & 0 & 0 & T_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ t_4 & 0 & 0 & 0 & T_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ t_5 & 0 & 1-T_5 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ t_6 & 0 & 0 & 0 & 0 & 0 & \frac{1}{T_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ t_7 & 0 & 0 & 0 & 1-T_7 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ t_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \frac{-1+T_8}{T_8} & 0 \\ t_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & T_{12} & 0 & 0 & 0 \\ t_{10} & 0 & 0 & 1-T_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ t_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{T_8} & 0 \\ t_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-T_{12} & 0 & 0 & 1 \end{pmatrix}$$

$\gamma // m_{14 \rightarrow 1} // m_{15 \rightarrow 1} // m_{16 \rightarrow 1} // m_{27 \rightarrow 2} // m_{28 \rightarrow 2} // m_{29 \rightarrow 2} // m_{2,10 \rightarrow 2} // m_{2,11 \rightarrow 2} // m_{3,12 \rightarrow 3}$

$$\begin{pmatrix} \frac{(-T_1-T_2+T_1 T_2) (-T_2-T_3+T_2 T_3)}{T_1 T_2} & h_1 & h_2 & h_3 \\ t_1 & -\frac{T_2}{-T_1-T_2+T_1 T_2} & \frac{-1+T_1}{-T_1-T_2+T_1 T_2} & 0 \\ t_2 & -\frac{T_1 (-1+T_2) T_3}{(-T_1-T_2+T_1 T_2) (-T_2-T_3+T_2 T_3)} & -\frac{(1-2 T_1-T_2+T_1 T_2) T_3}{(-T_1-T_2+T_1 T_2) (-T_2-T_3+T_2 T_3)} & \frac{-1+T_2}{-T_2-T_3+T_2 T_3} \\ t_3 & \frac{T_1 (-1+T_2) T_2 (-1+T_3)}{(-T_1-T_2+T_1 T_2) (-T_2-T_3+T_2 T_3)} & \frac{T_2 (1-2 T_1-T_2+T_1 T_2) (-1+T_3)}{(-T_1-T_2+T_1 T_2) (-T_2-T_3+T_2 T_3)} & \frac{1-2 T_2-T_3+T_2 T_3}{-T_2-T_3+T_2 T_3} \end{pmatrix}$$

$$\gamma = \text{Rp}_{16} \text{Rm}_{74} \text{Rm}_{52} \text{Rp}_{8,11} \text{Rm}_{12,9} \text{Rm}_{10,3};$$

$$\gamma // \text{m}_{14 \rightarrow 1} // \text{m}_{15 \rightarrow 1} // \text{m}_{16 \rightarrow 1} // \text{m}_{27 \rightarrow 2} // \text{m}_{28 \rightarrow 2} // \text{m}_{29 \rightarrow 2} // \text{m}_{2,10 \rightarrow 2} // \text{m}_{2,11 \rightarrow 2} // \text{m}_{3,12 \rightarrow 3}$$

$$\begin{pmatrix} \frac{(-1+T_1+T_2)(-1+T_2+T_3)}{T_2 T_3} & h_1 & h_2 & h_3 \\ t_1 & \frac{T_1}{-1+T_1+T_2} & \frac{(-1+T_1) T_2}{-1+T_1+T_2} & 0 \\ t_2 & \frac{(-1+T_2) T_2}{(-1+T_1+T_2)(-1+T_2+T_3)} & -\frac{T_2(1-T_1-2T_2+T_1 T_2)}{(-1+T_1+T_2)(-1+T_2+T_3)} & \frac{(-1+T_2) T_3}{-1+T_2+T_3} \\ t_3 & \frac{(-1+T_2)(-1+T_3)}{(-1+T_1+T_2)(-1+T_2+T_3)} & -\frac{(1-T_1-2T_2+T_1 T_2)(-1+T_3)}{(-1+T_1+T_2)(-1+T_2+T_3)} & -\frac{1-T_2-2T_3+T_2 T_3}{-1+T_2+T_3} \end{pmatrix}$$

$$\gamma = \text{Rm}_{16} \text{Rp}_{74} \text{Rp}_{52} \text{Rm}_{8,11} \text{Rp}_{12,9} \text{Rp}_{10,3};$$

$$\gamma // \text{m}_{41 \rightarrow 1} // \text{m}_{51 \rightarrow 1} // \text{m}_{61 \rightarrow 1} // \text{m}_{72 \rightarrow 2} // \text{m}_{82 \rightarrow 2} // \text{m}_{92 \rightarrow 2} // \text{m}_{10,2 \rightarrow 2} // \text{m}_{11,2 \rightarrow 2} // \text{m}_{12,3 \rightarrow 3}$$

$$\begin{pmatrix} (-T_1 - T_2 + T_1 T_2) & (-T_2 - T_3 + T_2 T_3) & h_1 & h_2 & h_3 \\ t_1 & & \frac{1-T_1-2T_2+T_1 T_2}{-T_1-T_2+T_1 T_2} & \frac{(-1+T_1) T_2 (1-T_2-2T_3+T_2 T_3)}{(-T_1-T_2+T_1 T_2)(-T_2-T_3+T_2 T_3)} & \frac{(-1+T_1)(-1+T_2) T_2 T_3}{(-T_1-T_2+T_1 T_2)(-T_2-T_3+T_2 T_3)} \\ t_2 & & \frac{-1+T_2}{-T_1-T_2+T_1 T_2} & -\frac{T_1(1-T_2-2T_3+T_2 T_3)}{(-T_1-T_2+T_1 T_2)(-T_2-T_3+T_2 T_3)} & -\frac{T_1(-1+T_2) T_3}{(-T_1-T_2+T_1 T_2)(-T_2-T_3+T_2 T_3)} \\ t_3 & & 0 & \frac{-1+T_3}{-T_2-T_3+T_2 T_3} & -\frac{T_2}{-T_2-T_3+T_2 T_3} \end{pmatrix}$$

$$\gamma = \text{Rp}_{16} \text{Rm}_{74} \text{Rm}_{52} \text{Rp}_{8,11} \text{Rm}_{12,9} \text{Rm}_{10,3};$$

$$\gamma_1 = \gamma // \text{m}_{41 \rightarrow 1} // \text{m}_{51 \rightarrow 1} // \text{m}_{61 \rightarrow 1} // \text{m}_{72 \rightarrow 2} // \text{m}_{82 \rightarrow 2} // \text{m}_{92 \rightarrow 2} // \text{m}_{10,2 \rightarrow 2} // \text{m}_{11,2 \rightarrow 2} // \text{m}_{12,3 \rightarrow 3}$$

$$\begin{pmatrix} \frac{(-1+T_1+T_2)(-1+T_2+T_3)}{T_1 T_2^2 T_3} & h_1 & h_2 & h_3 \\ t_1 & -\frac{1-2T_1-T_2+T_1 T_2}{-1+T_1+T_2} & -\frac{(-1+T_1)(1-2T_2-T_3+T_2 T_3)}{(-1+T_1+T_2)(-1+T_2+T_3)} & \frac{(-1+T_1)(-1+T_2)}{(-1+T_1+T_2)(-1+T_2+T_3)} \\ t_2 & \frac{T_1(-1+T_2)}{-1+T_1+T_2} & -\frac{T_2(1-2T_2-T_3+T_2 T_3)}{(-1+T_1+T_2)(-1+T_2+T_3)} & \frac{(-1+T_2) T_2}{(-1+T_1+T_2)(-1+T_2+T_3)} \\ t_3 & 0 & \frac{T_2(-1+T_3)}{-1+T_2+T_3} & \frac{T_3}{-1+T_2+T_3} \end{pmatrix}$$

$$\{\gamma_1 // \text{m}_{23 \rightarrow 2}, \gamma_1 // \text{m}_{32 \rightarrow 2}\}$$

$$\left\{ \begin{pmatrix} \frac{1-T_1-2T_2+2T_1 T_2+T_2^2}{T_1 T_2^2} & h_1 & h_2 \\ t_1 & -\frac{-1+T_1+2T_2-3T_1 T_2-T_2^2+T_1 T_2^2}{1-T_1-2T_2+2T_1 T_2+T_2^2} & -\frac{(-1+T_1)(1-3T_2+T_2^2)}{1-T_1-2T_2+2T_1 T_2+T_2^2} \\ t_2 & \frac{T_1(-1+T_2) T_2}{1-T_1-2T_2+2T_1 T_2+T_2^2} & \frac{T_2(1-T_1+T_1 T_2)}{1-T_1-2T_2+2T_1 T_2+T_2^2} \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} -\frac{(-1+T_1+T_2)(1-3T_2+T_2^2)}{T_1 T_2^2} & h_1 & h_2 \\ t_1 & -\frac{1-2T_1-T_2+T_1 T_2}{-1+T_1+T_2} & \frac{-1+T_1}{-1+T_1+T_2} \\ t_2 & \frac{T_1(-1+T_2)}{-1+T_1+T_2} & \frac{T_2}{-1+T_1+T_2} \end{pmatrix} \right\}$$

$$\begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \xi \end{pmatrix} = \begin{pmatrix} -\frac{1-2T_1-T_2+T_1 T_2}{-1+T_1+T_2} & -\frac{(-1+T_1)(1-2T_2-T_3+T_2 T_3)}{(-1+T_1+T_2)(-1+T_2+T_3)} & \frac{(-1+T_1)(-1+T_2)}{(-1+T_1+T_2)(-1+T_2+T_3)} \\ \frac{T_1(-1+T_2)}{-1+T_1+T_2} & -\frac{T_2(1-2T_2-T_3+T_2 T_3)}{(-1+T_1+T_2)(-1+T_2+T_3)} & \frac{(-1+T_2) T_2}{(-1+T_1+T_2)(-1+T_2+T_3)} \\ 0 & \frac{T_2(-1+T_3)}{-1+T_2+T_3} & \frac{T_3}{-1+T_2+T_3} \end{pmatrix};$$

$$\omega = \frac{(-1+T_1+T_2)(-1+T_2+T_3)}{T_1 T_2^2 T_3};$$

$-\beta (\epsilon + \theta) + \gamma (\epsilon + \theta) + \epsilon \phi - \theta \psi / . \mathbf{T}_3 \rightarrow \mathbf{T}_2 // \text{Simplify}$

$$\frac{(1 + \mathbf{T}_1 (-2 + \mathbf{T}_2)) (-1 + \mathbf{T}_2)}{(-1 + \mathbf{T}_1 + \mathbf{T}_2) (-1 + 2 \mathbf{T}_2)}$$

$$\left( \begin{array}{ccc} -\frac{1-2\mathbf{T}_1-\mathbf{T}_2+\mathbf{T}_1\mathbf{T}_2}{-1+\mathbf{T}_1+\mathbf{T}_2} & -\frac{(-1+\mathbf{T}_1)(1-2\mathbf{T}_2-\mathbf{T}_3+\mathbf{T}_2\mathbf{T}_3)}{(-1+\mathbf{T}_1+\mathbf{T}_2)(-1+\mathbf{T}_2+\mathbf{T}_3)} & \frac{(-1+\mathbf{T}_1)(-1+\mathbf{T}_2)}{(-1+\mathbf{T}_1+\mathbf{T}_2)(-1+\mathbf{T}_2+\mathbf{T}_3)} \\ \frac{\mathbf{T}_1(-1+\mathbf{T}_2)}{-1+\mathbf{T}_1+\mathbf{T}_2} & -\frac{\mathbf{T}_2(1-2\mathbf{T}_2-\mathbf{T}_3+\mathbf{T}_2\mathbf{T}_3)}{(-1+\mathbf{T}_1+\mathbf{T}_2)(-1+\mathbf{T}_2+\mathbf{T}_3)} & \frac{(-1+\mathbf{T}_2)\mathbf{T}_2}{(-1+\mathbf{T}_1+\mathbf{T}_2)(-1+\mathbf{T}_2+\mathbf{T}_3)} \\ 0 & \frac{\mathbf{T}_2(-1+\mathbf{T}_3)}{-1+\mathbf{T}_2+\mathbf{T}_3} & \frac{\mathbf{T}_3}{-1+\mathbf{T}_2+\mathbf{T}_3} \end{array} \right) \cdot \{1, 1, 1\} // \text{FullSimplify}$$

$$\left\{ \frac{3 - 2 \mathbf{T}_3 + \mathbf{T}_2 (-5 + \mathbf{T}_2 + 2 \mathbf{T}_3) - \mathbf{T}_1 (4 - 3 \mathbf{T}_3 + \mathbf{T}_2 (-6 + \mathbf{T}_2 + 2 \mathbf{T}_3))}{(-1 + \mathbf{T}_1 + \mathbf{T}_2) (-1 + \mathbf{T}_2 + \mathbf{T}_3)}, \frac{\mathbf{T}_1 (-1 + \mathbf{T}_2) (-1 + \mathbf{T}_2 + \mathbf{T}_3) + \mathbf{T}_2 (-2 - \mathbf{T}_2 (-3 + \mathbf{T}_3) + \mathbf{T}_3)}{(-1 + \mathbf{T}_1 + \mathbf{T}_2) (-1 + \mathbf{T}_2 + \mathbf{T}_3)}, \frac{\mathbf{T}_2 (-1 + \mathbf{T}_3) + \mathbf{T}_3}{-1 + \mathbf{T}_2 + \mathbf{T}_3} \right\}$$

$$\{1, 1, 1\} \cdot \left( \begin{array}{ccc} -\frac{1-2\mathbf{T}_1-\mathbf{T}_2+\mathbf{T}_1\mathbf{T}_2}{-1+\mathbf{T}_1+\mathbf{T}_2} & -\frac{(-1+\mathbf{T}_1)(1-2\mathbf{T}_2-\mathbf{T}_3+\mathbf{T}_2\mathbf{T}_3)}{(-1+\mathbf{T}_1+\mathbf{T}_2)(-1+\mathbf{T}_2+\mathbf{T}_3)} & \frac{(-1+\mathbf{T}_1)(-1+\mathbf{T}_2)}{(-1+\mathbf{T}_1+\mathbf{T}_2)(-1+\mathbf{T}_2+\mathbf{T}_3)} \\ \frac{\mathbf{T}_1(-1+\mathbf{T}_2)}{-1+\mathbf{T}_1+\mathbf{T}_2} & -\frac{\mathbf{T}_2(1-2\mathbf{T}_2-\mathbf{T}_3+\mathbf{T}_2\mathbf{T}_3)}{(-1+\mathbf{T}_1+\mathbf{T}_2)(-1+\mathbf{T}_2+\mathbf{T}_3)} & \frac{(-1+\mathbf{T}_2)\mathbf{T}_2}{(-1+\mathbf{T}_1+\mathbf{T}_2)(-1+\mathbf{T}_2+\mathbf{T}_3)} \\ 0 & \frac{\mathbf{T}_2(-1+\mathbf{T}_3)}{-1+\mathbf{T}_2+\mathbf{T}_3} & \frac{\mathbf{T}_3}{-1+\mathbf{T}_2+\mathbf{T}_3} \end{array} \right) // \text{FullSimplify}$$

$\{1, 1, 1\}$

$\text{Clear}[\alpha, \beta, \gamma, \delta, \theta, \epsilon, \phi, \psi, \Xi, \mu, \omega]; \Gamma[\omega, \{t_a, t_b, t_s\} \cdot \begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \Xi \end{pmatrix} \cdot \{h_a, h_b, h_s\}]$

$$\begin{pmatrix} \omega & h_a & h_b & h_s \\ t_a & \alpha & \beta & \theta \\ t_b & \gamma & \delta & \epsilon \\ t_s & \phi & \psi & \Xi \end{pmatrix}$$

$\Gamma[\omega, \{t_a, t_b, t_s\} \cdot \begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \Xi \end{pmatrix} \cdot \{h_a, h_b, h_s\}] // \text{m}_{ab \rightarrow c}$

$$\begin{pmatrix} -(-1 + \beta) \omega & h_c & h_s \\ t_c & \frac{-\gamma + \beta \gamma - \alpha \delta}{-1 + \beta} & \frac{-\epsilon + \beta \epsilon - \delta \theta}{-1 + \beta} \\ t_s & \frac{-\phi + \beta \phi - \alpha \psi}{-1 + \beta} & \frac{-\Xi + \beta \Xi - \theta \psi}{-1 + \beta} \end{pmatrix}$$

$\Gamma[\omega, \{t_a, t_b, t_s\} \cdot \begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \Xi \end{pmatrix} \cdot \{h_a, h_b, h_s\}] // \text{m}_{ba \rightarrow c}$

$$\begin{pmatrix} -(-1 + \gamma) \omega & h_c & h_s \\ t_c & \frac{-\beta + \beta \gamma - \alpha \delta}{-1 + \gamma} & \frac{-\alpha \epsilon - \theta + \gamma \theta}{-1 + \gamma} \\ t_s & \frac{-\delta \phi - \psi + \gamma \psi}{-1 + \gamma} & \frac{-\Xi + \gamma \Xi - \epsilon \phi}{-1 + \gamma} \end{pmatrix}$$

$$(1 - \gamma) \begin{pmatrix} \frac{-\beta + \beta \gamma - \alpha \delta}{-1 + \gamma} & \frac{-\alpha \epsilon - \theta + \gamma \theta}{-1 + \gamma} \\ \frac{-\delta \phi - \psi + \gamma \psi}{-1 + \gamma} & \frac{-\Xi + \gamma \Xi - \epsilon \phi}{-1 + \gamma} \end{pmatrix} // \text{Simplify} // \text{MatrixForm}$$

$$\begin{pmatrix} \beta - \beta \gamma + \alpha \delta & \alpha \epsilon + \theta - \gamma \theta \\ \delta \phi + \psi - \gamma \psi & \Xi - \gamma \Xi + \epsilon \phi \end{pmatrix}$$

$$\text{FullSimplify}[(\Xi + \epsilon \phi / (1 - \gamma) - 1) (1 - \gamma) - (\Xi + \theta \psi / (1 - \beta) - 1) (1 - \beta) /. \{\Xi \rightarrow 1 - \epsilon - \theta\}]$$

$$-\beta (\epsilon + \theta) + \gamma (\epsilon + \theta) + \epsilon \phi - \theta \psi$$

$$\text{FullSimplify}[(\Xi + \epsilon \phi / (1 - \gamma) - 1) - (\Xi + \theta \psi / (1 - \beta) - 1) /. \{\Xi \rightarrow 1 - \epsilon - \theta\}]$$

$$\frac{\epsilon \phi}{1 - \gamma} + \frac{\theta \psi}{-1 + \beta}$$

```
ΓZ[L_] := Module[{s, Z, c, k},
  s = Skeleton[L];
  Z =
    Times@@PD[L] /. X[i_, j_, k_, l_] => If[PositiveQ[X[i, j, k, l]], Rp[i], Rm[j]];
  Do[Z = Z // m_s[[c,1]], s[[c,k]] -> s[[c,1]], {c, Length[s]}, {k, 2, Length[s[[c]]]};
  Z];
```

```
ΓA[K_] := ΓZ[K][[1]] /. T_ -> T;
```

```
ΓMVA[L_Link] := Module[{Hs, ω, μ, A},
  {ω, μ} = List @@ ΓZ[L];
  Hs = Rest[h_# & /@ (First /@ Skeleton[L])];
  A = Outer[Coefficient[μ, #1 * #2] &, Hs, Hs /. h_a_ -> t_a];
  Factor[
$$\frac{\omega \text{Det}[A - \text{IdentityMatrix}@\text{Length}@\text{Hs}]}{1 - T_{\text{Skeleton}[L][[1,1] ]}}$$

]
```

```
ΓZ[Link["L6a4"]]
```

KnofTheory::loading : Loading precomputed data in PD4Links`.

$$\begin{pmatrix} \frac{(1 - T_1 - T_5 + T_1 T_5 - T_9 + T_1 T_9 + T_5 T_9) (T_1 + T_5 - T_1 T_5 + T_9 - T_1 T_9 - T_5 T_9 + T_1 T_5 T_9)}{T_1 T_5 T_9} & h_1 \\ t_1 & \frac{T_9 (1 - 2 T_1 + T_1^2 - T_5 + 3 T_1 T_5 - T_1^2 T_5 - T_9 + 2 T_1 T_9 - T_1^2 T_9 + T_5 T_9 - 2 T_1 T_5 T_9 + (1 - T_1 - T_5 + T_1 T_5 - T_9 + T_1 T_9 + T_5 T_9) (T_1 + T_5 - T_1 T_5 + T_9 - T_1 T_9 - T_5 T_9 + T_1 T_5 T_9))}{(1 - T_1 - T_5 + T_1 T_5 - T_9 + T_1 T_9 + T_5 T_9) (T_1 + T_5 - T_1 T_5 + T_9 - T_1 T_9 - T_5 T_9 + T_1 T_5 T_9)} \\ t_5 & \frac{T_1 (-1 + T_5) (1 - T_1 + T_1 T_5) (-1 + T_9)}{(1 - T_1 - T_5 + T_1 T_5 - T_9 + T_1 T_9 + T_5 T_9) (T_1 + T_5 - T_1 T_5 + T_9 - T_1 T_9 - T_5 T_9 + T_1 T_5 T_9)} \\ t_9 & \frac{(-1 + T_5) T_5 (-1 + T_9) (1 - 2 T_1 - T_9 + T_1 T_9)}{(1 - T_1 - T_5 + T_1 T_5 - T_9 + T_1 T_9 + T_5 T_9) (T_1 + T_5 - T_1 T_5 + T_9 - T_1 T_9 - T_5 T_9 + T_1 T_5 T_9)} \end{pmatrix}$$

```
ΓA[Knof[8, 17]]
```

KnofTheory::loading : Loading precomputed data in PD4Knots`.

$$\frac{1 - 4 T + 8 T^2 - 11 T^3 + 8 T^4 - 4 T^5 + T^6}{T^2}$$

**FMVA**[[Link\["L6a4"\]](#)]

$$\frac{(-1 + T_1) (-1 + T_5) (-1 + T_9)}{T_1 T_5}$$

**Factor** $\left[\frac{\mathbf{Alexander}[\#][T]}{\Gamma\mathbf{A}[\#]}\right]$  & **/@ AllKnots**[{3, 9}]

$$\left\{ T, \frac{1}{T}, T^2, T^2, 1, 1, 1, T^3, T^3, \frac{1}{T^4}, \frac{1}{T^4}, T^3, T, \frac{1}{T^2}, T, T, \frac{1}{T}, \frac{1}{T}, \frac{1}{T^3}, T, \frac{1}{T}, \frac{1}{T}, \frac{1}{T}, T, \frac{1}{T}, \frac{1}{T}, T, T^3, T, \frac{1}{T}, 1, \frac{1}{T^4}, 1, T, T^4, T^4, \frac{1}{T^5}, T^4, \frac{1}{T^5}, T^4, T^4, 1, T^4, \frac{1}{T^5}, \frac{1}{T^3}, T^2, \frac{1}{T^5}, \frac{1}{T^3}, \frac{1}{T^3}, \frac{1}{T^5}, \frac{1}{T}, T^4, \frac{1}{T}, T^2, \frac{1}{T^3}, \frac{1}{T^2}, T^4, 1, T^2, \frac{1}{T^3}, T, T, \frac{1}{T}, 1, T, \frac{1}{T^3}, 1, 1, T^4, \frac{1}{T^3}, \frac{1}{T}, T^4, \frac{1}{T^4}, 1, 1, \frac{1}{T^2}, \frac{1}{T^3}, 1, T^2, 1, \frac{1}{T^2}, \frac{1}{T^4}, \frac{1}{T^6} \right\}$$

**Factor** $\left[\frac{1}{\Gamma\mathbf{MVA}[\#]}(\mathbf{MultivariableAlexander}[\#][T] /. \mathbf{T}[i_] \Rightarrow \mathbf{T}_{\mathbf{skeleton}[\#][i,1]})\right]$  & **/@ AllLinks**[{2, 8}]

**KnotTheory**:loading : Loading precomputed data in **MultivariableAlexander4Links**.

$$\left\{ -T_1^2 T_3, -T_1^{3/2} T_5^{3/2}, -\sqrt{T_1} T_5^{3/2}, -T_1^{3/2} \sqrt{T_5}, -T_1^2 T_7^2, -T_1 T_7^2, -\frac{\sqrt{T_1} \sqrt{T_5}}{\sqrt{T_9}}, -T_1^{3/2} T_5^{3/2} T_9^{3/2}, -\frac{\sqrt{T_1} \sqrt{T_5}}{T_9^{3/2}}, -\sqrt{T_1} \sqrt{T_5}, -T_1^{3/2} T_5^{7/2}, -\frac{\sqrt{T_1}}{T_5^{3/2}}, -\frac{\sqrt{T_1}}{T_5^{3/2}}, -T_1 T_7^2, -\frac{1}{T_7}, -\frac{T_1^{3/2} \sqrt{T_5}}{\sqrt{T_9}}, -T_1^{3/2} T_5^{7/2}, -\sqrt{T_1} T_5^{5/2}, -\sqrt{T_1} T_5^{3/2}, -\frac{\sqrt{T_1}}{\sqrt{T_5}}, -T_1^{3/2} T_5^{3/2}, -\sqrt{T_1} T_5^{3/2}, -\frac{T_1^{3/2}}{\sqrt{T_5}}, -\frac{T_1^{3/2}}{\sqrt{T_5}}, -T_1^{3/2} T_5^{7/2}, -\frac{T_1}{T_7}, -T_1 T_7, -T_1^2 T_7^3, -T_1^2 T_7^3, -T_1^{5/2} T_9^{5/2}, -T_1^{5/2} T_9^{5/2}, -T_1^{5/2} T_9^{5/2}, -T_1^{3/2} T_5^{3/2} \sqrt{T_9}, -\sqrt{T_1}, -T_1^{3/2} T_5^2 T_{11}^2, -\frac{\sqrt{T_1}}{T_{11}^2}, -\frac{\sqrt{T_1} T_5}{T_{11}}, -\frac{T_1^{3/2} \sqrt{T_5}}{T_{13}^{3/2}}, -T_1^{3/2} T_5^{3/2} T_9^{3/2} T_{13}^{3/2}, -T_1^{3/2} T_5^{3/2}, -\sqrt{T_1} \sqrt{T_5}, -T_1^{3/2} T_5^2 T_{11}^2, -\sqrt{T_1} T_5^2 T_{11}, -\sqrt{T_1} T_5^{3/2} T_{11}^{3/2}, -T_1^{3/2} T_5^{5/2} \sqrt{T_{13}}, -\frac{\sqrt{T_1} \sqrt{T_5}}{\sqrt{T_9} \sqrt{T_{13}}}, -\sqrt{T_1} \sqrt{T_5} \sqrt{T_9} \sqrt{T_{13}} \right\}$$