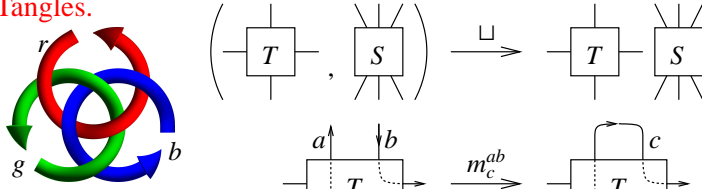


Abstract. I will describe some very good formulas for a (matrix plus scalar)-valued extension of the Alexander polynomial to tangles, then say that everything extends to virtual tangles, then roughly to simply knotted balloons and hoops in 4D, then the target space extends to (free Lie algebras plus cyclic words), and the result is a universal finite type of the knotted objects in its domain. Taking a cue from the BF topological quantum field theory, everything should extend (with some modifications) to arbitrary codimension-2 knots in arbitrary dimension and in particular, to arbitrary 2-knots in 4D. But what is really going on is still a mystery.

Tangles.



Why Tangles?

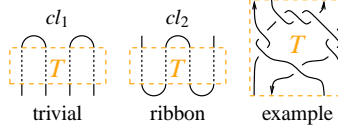
• Finitely presented. (meta-associativity: m_a^{ab} // m_a^{ac} = m_b^{bc} // m_a^{ab})

• Divide and conquer proofs and computations.

• "Algebraic Knot Theory": If K is ribbon,

Z(K) ∈ {cl_2(Z): cl_1(Z) = 1}.

(Genus and crossing number are also definable properties).



Theorem 1. ∃! an invariant γ: {pure framed S-component tangles} → R × M_{S×S}(R), where R = R_S = ℤ((T_a)_{a∈S}) is the ring of rational functions in S variables, intertwining

1. (ω_1 | S_1, ω_2 | S_2) → (ω_1 ω_2 | S_1 S_2, S_1 A_1 0, S_2 0 A_2)

2. (ω | a b S) → (μ ω | c S, c γ + α δ / μ, ε + δ θ / μ, S φ + α ψ / μ, Ξ + ψ θ / μ)_{T_a, T_b → T_c}

and satisfying (|a; a ↗ b, b ↘ a) → (1 | a; a 1 1 - T_a^{±1}, a | 0 T_a^{±1})

In Addition, • This is really "just" a stitching formula for Burau/Gassner [LD, KLW, CT].

- L ↦ ω is Alexander, mod units.
• L ↦ (ω, A) ↦ ω det'(A - I)/(1 - T') is the MVA, mod units.
• The "fastest" Alexander algorithm.
• There are also formulas for strand deletion, reversal, and doubling.
• Every step along the computation is the invariant of something.
• Extends to and more naturally defined on v/w-tangles.
• Fits in one column, including propaganda & implementation.



Implementation key idea:

(ω, A = (α_{ab})) ↔ (ω, λ = ∑ α_{ab} t_a h_b)
F := F[ω1, λ1] F[ω2, λ2] := F[ω1 ω2, λ1 λ2];
m_{a,b} := F[ω, λ] := Module[(α, β, γ, δ, θ, ε, φ, ψ, Ξ, μ),
(α β θ)
(γ δ ε)
(φ ψ Ξ) := (∂_{t_a, h_a} λ ∂_{t_b, h_b} λ ∂_{t_c, λ} / (∂_{h_a, λ} ∂_{h_b, λ} λ)) / (t | h)_{a1b} → 0;
F[(μ = 1 - β) ω, {t_c, 1}. (γ + α δ / μ, ε + δ θ / μ, φ + α ψ / μ, Ξ + ψ θ / μ). {h_a, 1}] / . {T_a → T_c, T_b → T_c} // FCollect];
FP_{a,b} := F[1, {t_a, t_b}. (1 1 - T_a) / T_a]. {h_a, h_b};
RM_{a,b} := RP_{ab} / . T_a → 1 / T_a;

Meta-Associativity

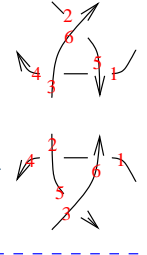
γ = F[ω, {t_1, t_2, t_3, t_s}. (α_{11} α_{12} α_{13} θ_1, α_{21} α_{22} α_{23} θ_2, α_{31} α_{32} α_{33} θ_3, φ_1 φ_2 φ_3 Ξ) . {h_1, h_2, h_3, h_s}];

(γ // m_{12→1} // m_{13→1}) = (γ // m_{23→2} // m_{12→1})

True R3 ... divide and conquer!

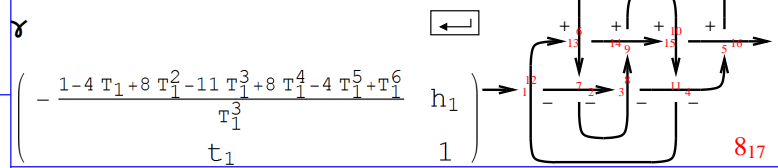
{Rm_{51} Rm_{62} Rp_{34} // m_{14→1} // m_{25→2} // m_{36→3}, Rp_{61} Rm_{24} Rm_{35} // m_{14→1} // m_{25→2} // m_{36→3}}

(1 h_1 h_2 h_3, t_1 T_3 T_2 0 0, t_2 -1+T_2 T_2 1 T_3 0, t_3 -1+T_3 T_2 -1+T_3 1) (1 h_1 h_2 h_3, t_1 T_3 T_2 0 0, t_2 -1+T_2 T_2 1 T_3 0, t_3 -1+T_3 T_2 -1+T_3 1)



γ = Rm_{12,1} Rm_{27} Rm_{83} Rm_{4,11} Rp_{16,5} Rp_{6,13} Rp_{14,9} Rp_{10,15};

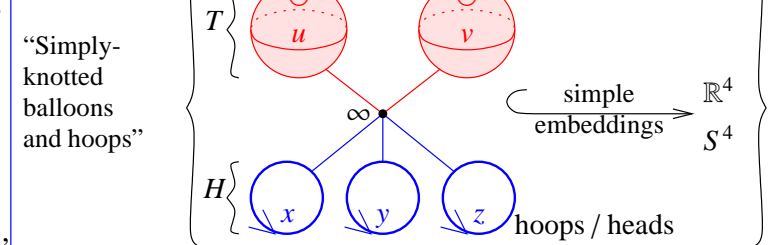
Do [γ = γ // m_{1k→1}, {k, 2, 16}];



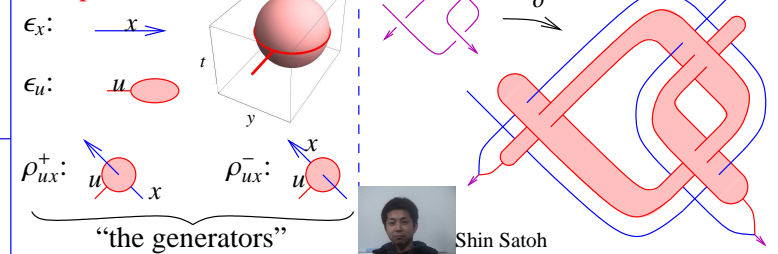
Weaknesses, • m_c^{ab} is non-linear.

• The product ωA is always Laurent, but proving this takes induction with exponentially many conditions.

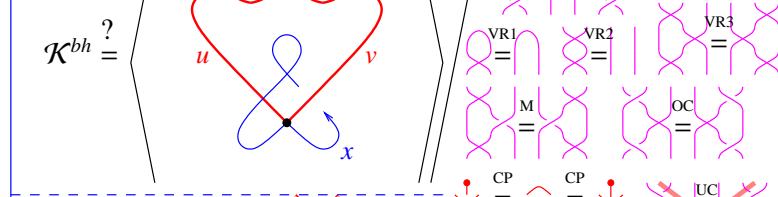
K^{bh}(H; T).



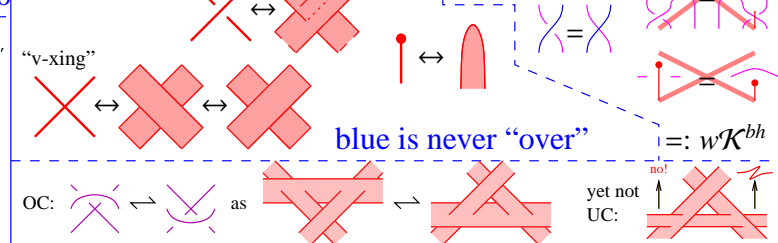
Examples.



Disturbing Conjecture

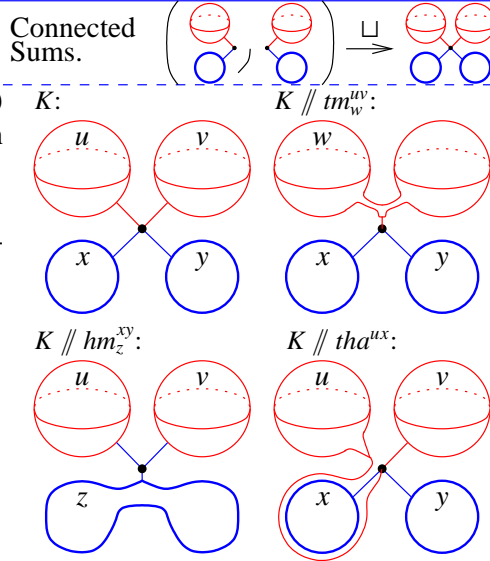


Dictionary.



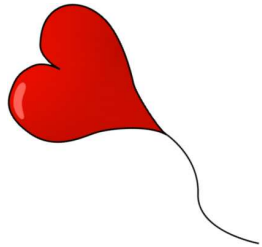
Operations

Punctures & Cuts



If X is a space, $\pi_1(X)$ is a group, $\pi_2(X)$ is an Abelian group, and π_1 acts on π_2 .

Proposition. The generators generate.



Definition. l_{xu} is the linking number of hoop x with balloon u . For $x \in H$, $\sigma_x := \prod_{u \in T} T_u^{l_{xu}} \in R = R_T = \mathbb{Z}((T_a)_{a \in T})$, the ring of rational functions in T variables.

Theorem 2 [BNS]. $\exists!$ an invariant $\beta: w\mathcal{K}^{bh}(H; T) \rightarrow R \times M_{T \times H}(R)$, intertwining

$$1. \left(\begin{array}{c|c} \omega_1 & H_1 \\ \hline T_1 & A_1 \end{array}, \begin{array}{c|c} \omega_2 & H_2 \\ \hline T_2 & A_2 \end{array} \right) \xrightarrow{\sqcup} \begin{array}{c|cc} \omega_1 \omega_2 & H_1 & H_2 \\ \hline T_1 & A_1 & 0 \\ \hline & 0 & A_2 \end{array}$$

$$2. \begin{array}{c|c} \omega & H \\ \hline u & \alpha \\ v & \beta \\ T & \Xi \end{array} \xrightarrow{tm_w^{uv}} \begin{array}{c|c} \omega & H \\ \hline w & \alpha + \beta \\ T & \Xi \end{array}_{T_u, T_v \rightarrow T_w}$$

$$3. \begin{array}{c|cc} \omega & x & y & H \\ \hline T & \alpha & \beta & \Xi \end{array} \xrightarrow{hm_z^{xy}} \begin{array}{c|c} \omega & z & H \\ \hline T & \alpha + \sigma_x \beta & \Xi \end{array}$$

$$4. \begin{array}{c|cc} \omega & x & H \\ \hline u & \alpha & \theta \\ T & \phi & \Xi \end{array} \xrightarrow[\nu:=1+\alpha]{tha^{ux}} \begin{array}{c|cc} \nu\omega & x & H \\ \hline u & \sigma_x \alpha / \nu & \sigma_x \theta / \nu \\ T & \phi / \nu & \Xi - \phi \theta / \nu \end{array}$$

and satisfying $(\epsilon_x; \epsilon_u; \rho_{ux}^\pm) \xrightarrow{\beta} \left(\frac{1}{u} \middle| x; \frac{1}{u} \middle| ; \frac{1}{u} \middle| T_u^{\pm 1} - 1 \right)$.

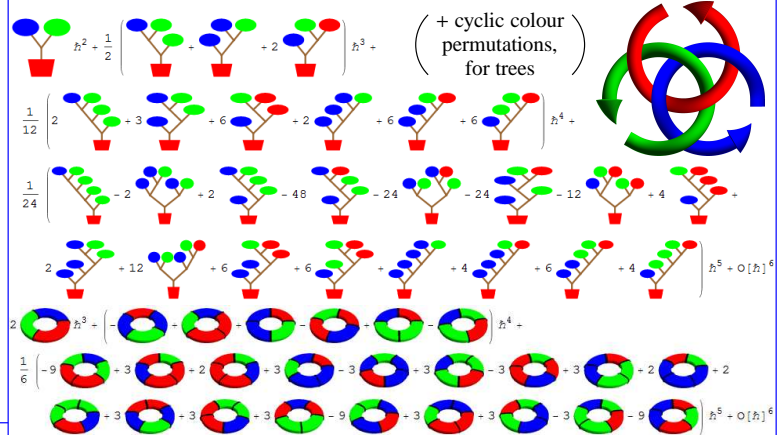
Proposition. If T is a u-tangle and $\beta(\delta T) = (\omega, A)$, then $\gamma(T) = (\omega, \sigma - A)$, where $\sigma = \text{diag}(\sigma_a)_{a \in S}$. Under this, $m_c^{ab} \leftrightarrow tha^{ab} // tm_c^{ab} // hm_c^{ab}$.

References.

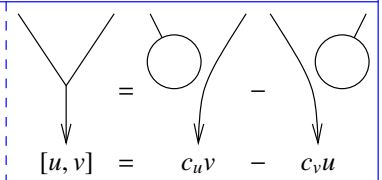
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Theorem 3 [BND, BN]. $\exists!$ a homomorphic expansion, aka a homomorphic universal finite type invariant Z of w -knotted balloons and hoops. $\zeta := \log Z$ takes values in $FL(T)^H \times CW(T)$.

ζ is computable! ζ of the Borromean tangle, to degree 5:



Proposition [BN]. Modulo all relations that universally hold for the 2D non-Abelian Lie algebra and after some changes-of-variable, ζ reduces to β and the KBH operations on ζ reduce to the formulas in Theorem 2.



A Big Question. Does it all extend to arbitrary 2-knots (not necessarily “simple”)? To arbitrary codimension-2 knots?

BF Following [CR]. $A \in \Omega^1(M = \mathbb{R}^4, \mathfrak{g})$, $B \in \Omega^2(M, \mathfrak{g}^*)$,

$$S(A, B) := \int_M \langle B, F_A \rangle.$$

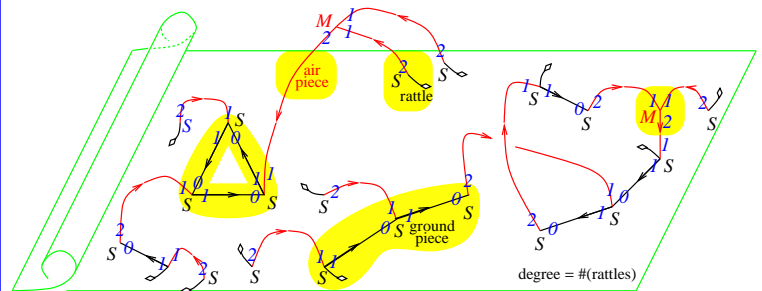
With $\kappa: (S = \mathbb{R}^2) \rightarrow M$, $\beta \in \Omega^0(S, \mathfrak{g})$, $\alpha \in \Omega^1(S, \mathfrak{g}^*)$, set

$$O(A, B, \kappa) := \int \mathcal{D}\beta \mathcal{D}\alpha \exp\left(\frac{i}{\hbar} \int_S \langle \beta, d_{\kappa^*} \alpha + \kappa^* B \rangle\right).$$

The BF Feynman Rules. For an edge e , let Φ_e be its direction, in S^3 or S^1 . Let ω_3 and ω_1 be volume forms on S^3 and S^1 . Then

$$Z_{BF} = \sum_{\text{diagrams } D} \frac{|D|}{|\text{Aut}(D)|} \int_{\mathbb{R}^2} \dots \int_{\mathbb{R}^2} \int_{\mathbb{R}^4} \dots \int_{\mathbb{R}^4} \prod_{\text{red } e \in D} \Phi_e^* \omega_3 \prod_{\text{black } e \in D} \Phi_e^* \omega_1$$

(modulo some STU - and IHX -like relations).



Issues. • Signs don't quite work out, and BF seems to reproduce only “half” of the wheels invariant.

- There are many more configuration space integrals than BF Feynman diagrams and than just trees and wheels.
- I don't know how to define “finite type” for arbitrary 2-knots.

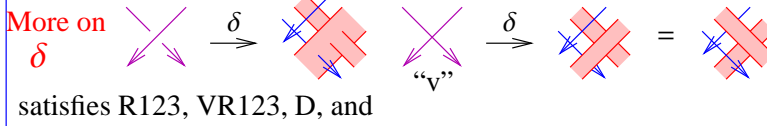
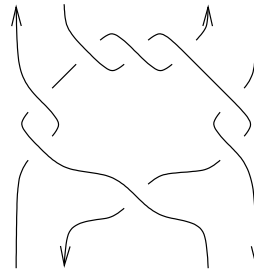
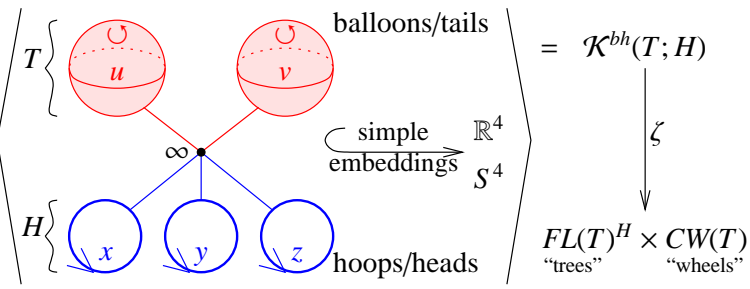


“God created the knots, all else in topology is the work of mortals.”

Leopold Kronecker (modified)



Safekeeping / Recycling.



- δ injects u-knots into \mathcal{K}^{bh} (likely u-tangles too).
- δ maps v-tangles to \mathcal{K}^{bh} ; the kernel contains the above and **conjecturally** (Satoh), that's all.
- Allowing punctures and cuts, δ is onto.