I will describe a semi-rigorous reduction of perturbative BF theory (Cattaneo-Rossi [CR]) to computable combinatorics, in the case of ribbon 2-links. Also, I will explain how and why my approach may or may not work in the non-ribbon case. Weak this result is, and at least partially already known (Watanabe [Wa]). Yet in the ribbon case, the resulting invariant is a universal finite type invariant, a gadget that significantly generalizes and clarifies the Alexander polynomial and that is closely related to the Kashiwara-Vergne problem. I cannot rule out the possibility that the corresponding gadget in the non-ribbon case will be as interesting.

(1) BF Following [CR]. $A \in \Omega^1(M = \mathbb{R}^4, \mathfrak{g}), B \in \Omega^2(M, \mathfrak{g}^*)$,

\[ S(A, B) := \int_M \langle B, F_A \rangle. \]

With $\kappa: (S = \mathbb{R}^2) \to M, \beta, \alpha \in \Omega^1(S, \mathfrak{g})$, set

\[ O(A, B, \kappa) := \int D\beta D\alpha \exp \left( \frac{i}{\hbar} \int_S \langle \beta, d_{\kappa^*} \alpha + \kappa^* B \rangle \right). \]

Decker Sets (“2D Gauss Codes”):

- “a double curve”
- “a triple point”
- “a branch point”

Some Examples.

- A 4D knot by Carter and Saito [CS]
- A 4D knot by Dalvit [Da]
- A 2-link

The BF Feynman Rules.

- For an edge $e$, let $\Phi_e$ be its direction, in $S^3$ or $S^1$. Let $\omega_3$ and $\omega_1$ be volume forms on $S^3$ and $S^1$. Then for a 2-link $(\kappa_e)_{e \in T}$,

\[ \zeta = \log \frac{\sum_{\text{diagrams } D} |\text{Aut}(D)|}{\text{dim}(D)} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \prod_{\text{red } e \in D} \Phi_{\kappa_e}^* \omega_3 \prod_{\text{black } e \in D} \Phi_{\kappa_e}^* \omega_1 \]

is an invariant in $CW(FL(T)) \to CW(T)/\sim$, “symmetrized cyclic words in $T$”.

A BF Feynman Diagram.
Theorem 1 (with Cattaneo, Dalvit (credit, no blame)). In the ribbon case, $e^\ell$ can be computed as follows:

\[
\sum_{k,l,m \geq 0} \frac{(\pm)^{k+l+m}k!l!m!}{k!l!m!}
\]

of course also fits, hence contains the MV $A$.

In 3D, one can’t zoom in and compute “the Alexander.”

In 3D, a generic immersion of $\gamma$ also gives rise to a knot. In 4D, a generic immersion of a surface has finitely-many double points (a “gnot”?)

\[\text{References.}
\]


Continuing Joost Slingerland...

Sketch of Proof. In 4D, the “true” triple linking number

http://drorbn.net/AcademicPensieve/2014-05/4DC

Is this all? What about the $v$-invariant?

There’s an alternative definition of finite type in 4D, due to Goussarov (see [BN1]). The obvious parallel in 4D involves “bubble wraps”. Is it any good?

Finite type. What are finite-type invariants for 2-knots? What would be “chord diagrams”?

Bubble-wrap-finite-type.

There’s an alternative definition of finite type in 3D, due to Goussarov (see [BN1]). The obvious parallel in 4D involves “bubble wraps”. Is it any good?

Shielded tangles. In 3D, one can’t zoom in and compute “the Chern-Simons invariant of a tangle”. Yet there are well-defined invariants of “shielded tangles”, and rules for their compositions. What would the 4D analog be?

Will the relationship with the Kashivara-Vergne problem [BND] necessarily arise here?

Plane curves. Shouldn’t we understand integral / finite type invariants of plane curves, in the style of Arnold’s $J^+$, $J^-$, and $Sf$ [Ar], a bit better?

J^+ 0 0 2 0 0 0 0 3 0
J^- 0 0 0 0 2 0 0 0 3
Sf 0 0 0 0 0 0 0 0

“God created the knots, all else in topology is the work of mortals.”

Leopold Kronecker (modified)
A 4DC Discussion.

Definition. A \(d\)-dimensional balanced diagram is a red-black S&M diagram whose half-edges can be weighted with non-negative integers so that the weight of red edges is \(d-1\), the weight of black edges is \(d-3\), the weight of \(M\) vertices is \(d\) and the weight of \(M\) vertices is \(d-2\).

Question. What are all \(d\)-dimensional balanced diagramss?
Safekeeping / Recycling.

manx

tail

air

piece

degree = \#(rattles)

even rattles

trees on wheels,
odd edges

ground

piece

manx
tail

degree = \#(manx tails)

k

l

m

degree = \#(rattles)

k

m

l

rattles

airpiece

snake

rattles

airpiece