A 4D Challenge, 1

Abstract. I will describe a semi-rigorous reduction of perturbative BF theory (Cattaneo-Rossi [CR]) to computable combinatorics, in the case of ribbon 2-links. Also, I will explain how and why my approach may or may not work in the non-ribbon case. Weak this result is, and at least partially already known (Watanabe [Wa]). Yet in the ribbon case, the resulting invariant is a universal finite type invariant, a gadget that significantly generalizes and clarifies the Alexander polynomial and that is closely related to the Kashiwara-Vergne problem. I cannot rule out the possibility that the corresponding gadget in the non-ribbon case will be as interesting. (good news in highlight)

BF Following [CR]. $A \in \Omega^1(M = \mathbb{R}^4, \mathfrak{g})$, $B \in \Omega^2(M, \mathfrak{g}^*)$,

$$S(A,B) := \int_M \langle B, F_A \rangle.$$

With $\kappa\colon (S = \mathbb{R}^2) \to M$, $\beta \in \Omega^0(S, \mathfrak{g})$, $\alpha \in \Omega^1(S, \mathfrak{g}^*)$, set

$$O(A,B,\kappa) := \int \mathcal{D}\beta\mathcal{D}\alpha \exp\left(\frac{i}{\hbar}\int_S \langle \beta, d_{\kappa^* A}\alpha + \kappa^* B\rangle\right).$$

The BF Feynman Rules. For an edge e, let Φ_e be its direction, in S^3 or S^1. Let ω_3 and ω_1 be volume forms on S^3 and S_1. Then for a 2-link $(\kappa_t)_{t\in T}$,

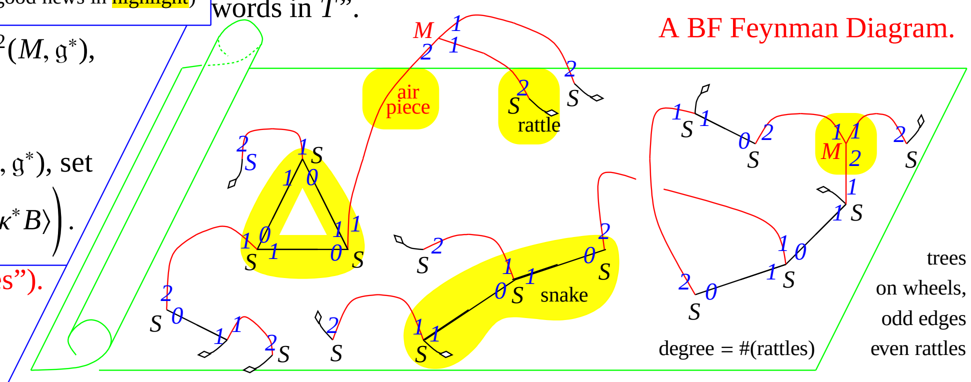
Cattaneo

Rossi

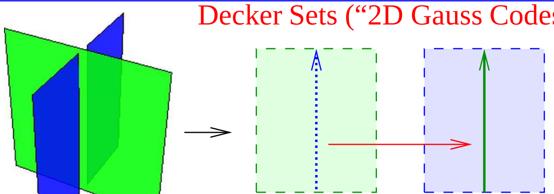
$$\zeta = \log \sum_{\substack{\text{diagrams}\\ D}} \frac{[D]}{|\mathrm{Aut}(D)|} \underbrace{\int_{\mathbb{R}^2}\cdots\int_{\mathbb{R}^2}}_{S\text{-vertices}} \underbrace{\int_{\mathbb{R}^4}\cdots\int_{\mathbb{R}^4}}_{M\text{-vertices}} \prod_{\substack{\text{red}\\ e\in D}} \Phi_e^*\omega_3 \prod_{\substack{\text{black}\\ e\in D}} \Phi_e^*\omega_1$$

is an invariant in $CW(FL(T)) \to CW(T)/\sim$, "symmetrized cyclic words in T".

A BF Feynman Diagram.

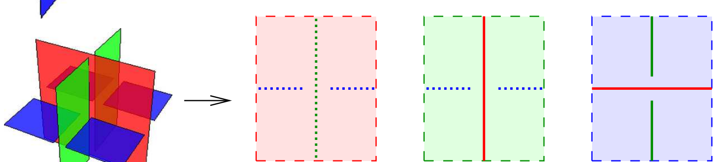


Decker Sets ("2D Gauss Codes").

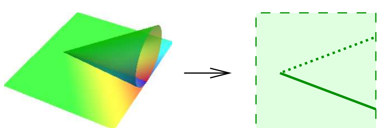


"a double curve"

(only double curves are allowed in ribbon 2-knots)

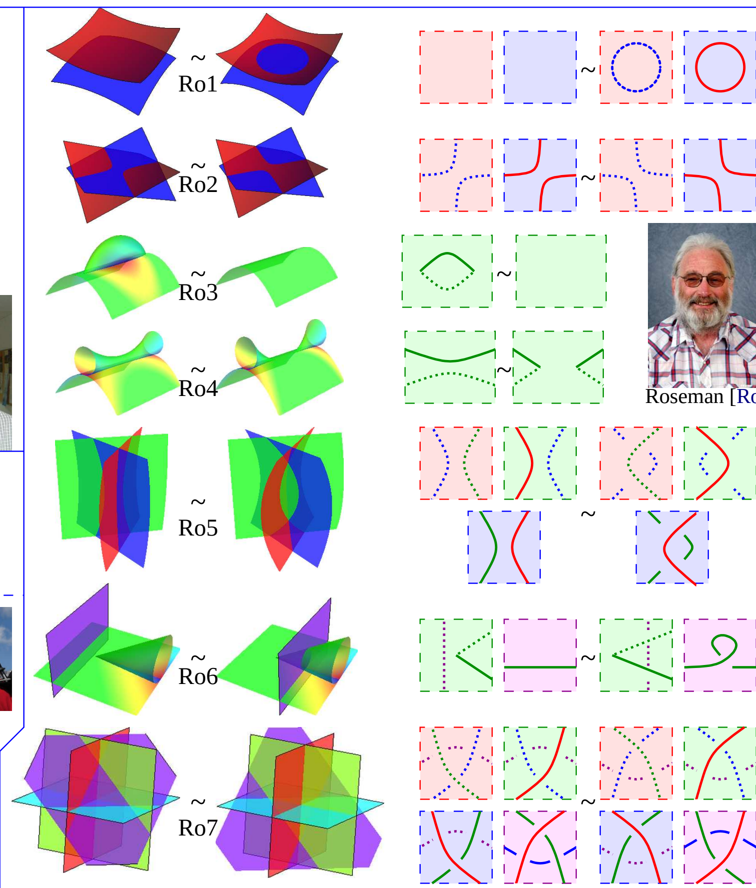


"a triple point"



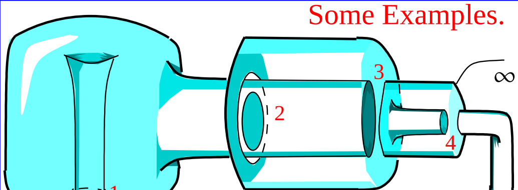
"a branch point"

Carter
Banach
Saito

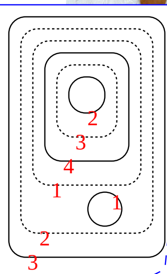


Roseman [Ro]

Some Examples.

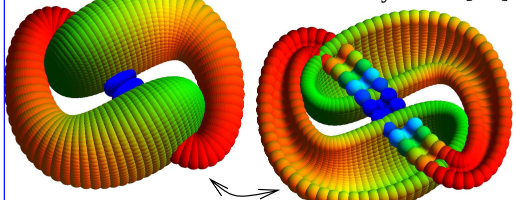


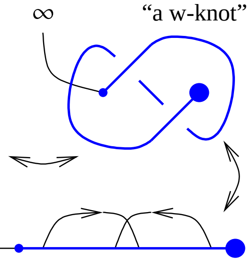
A 4D knot by Carter and Saito [CS]

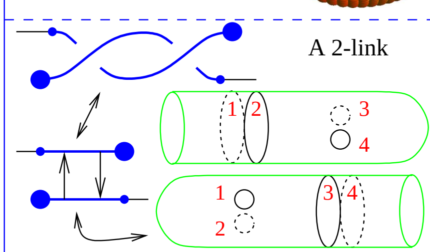


Dalvit

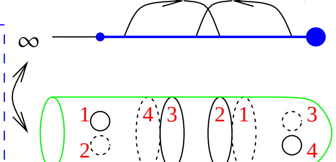
A 4D knot by Dalvit [Da]



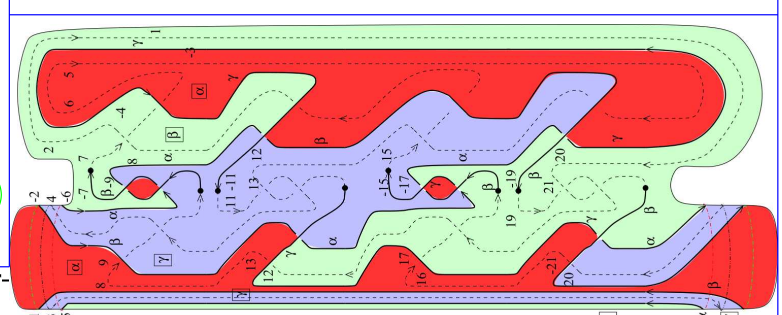




A 2-link



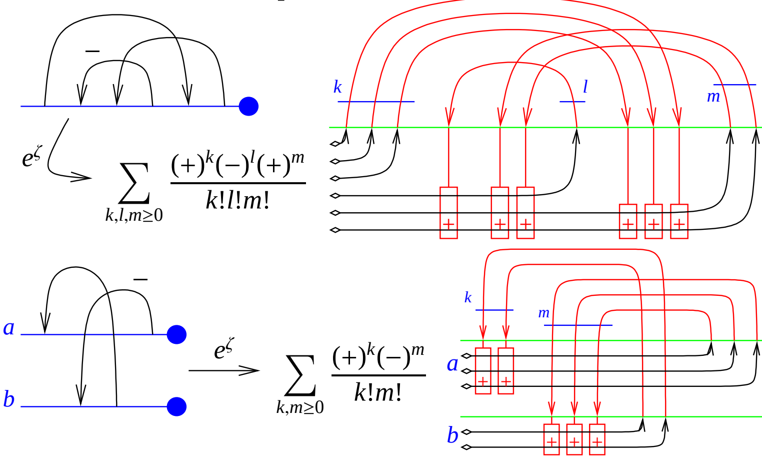
"ribbed cigar presentation"

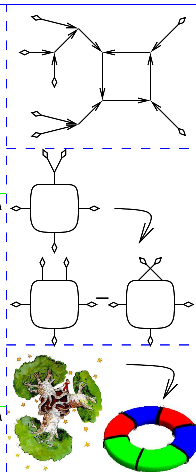


A 2-twist spun trefoil by Carter-Kamada-Saito [CKS].

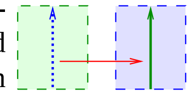
A 4D Challenge, 2

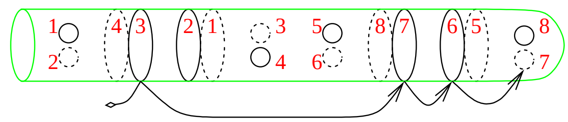
Theorem 1 (with Cattaneo, Dalvit (credit, no blame)). In the ribbon case, e^ζ can be computed as follows:

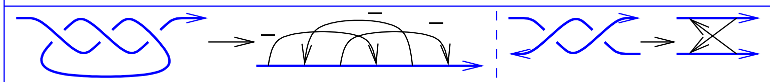




Sketch of Proof. In $4D$ axial gauge, only "drop down" red propagators, hence in the ribbon case, no M-trivalent vertices. S integrals are ± 1 iff "ground pieces" run on nested curves as below, and exponentials arise when several propagators compete for the same double curve. And then the combinatorics is obvious. . .





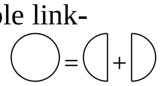


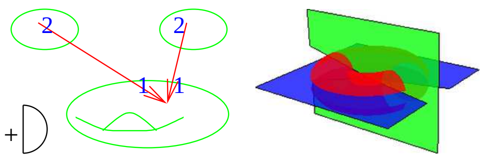
Theorem 2. Using Gauss diagrams to represent knots and T-component pure tangles, the above formulas define an invariant in $CW(FL(T)) \to CW(T)$, "cyclic words in T".
• Agrees with BN-Dancso [BND] and with [BN2]. • In-practice computable! • Vanishes on braids. • Extends to w. • Contains Alexander. • The "missing factor" in Levine's factorization [Le] (the rest of [Le] also fits, hence contains the MVA). • Related to / extends Farber's [Fa]? • Should be summed and categorified.

Musings

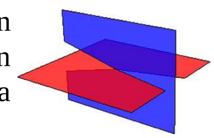
Chern-Simons. When the domain of BF is restricted to ribbon knots, and the target of Chern-Simons is restricted to trees and wheels, they agree. Why?

Is this all? What about the ∨-invariant? (the "true" triple linking number)

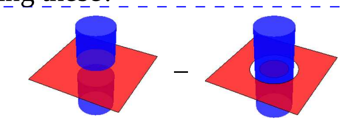




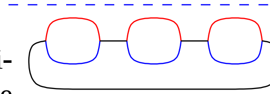
Gnots. In 3D, a generic immersion of S^1 is an embedding, a knot. In 4D, a generic immersion of a surface has finitely-many double points (a gnot?). Perhaps we should be studying these?



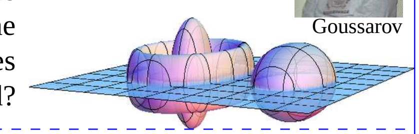
Finite type. What are finite-type invariants for 2-knots? What would be "chord diagrams"?



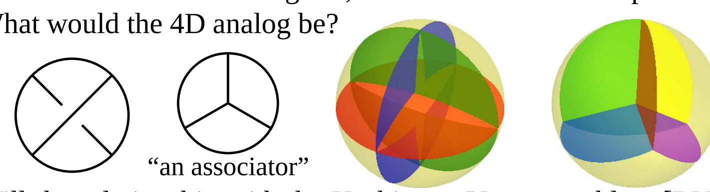
Bubble-wrap-finite-type. There's an alternative definition of finite type in 3D, due to Goussarov (see [BN1]). The obvious parallel in 4D involves "bubble wraps". Is it any good?




Goussarov

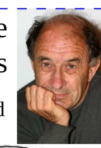


Shielded tangles. In 3D, one can't zoom in and compute "the Chern-Simons invariant of a tangle". Yet there are well-defined invariants of "shielded tangles", and rules for their compositions. What would the 4D analog be?



Will the relationship with the Kashiwara-Vergne problem [BND] necessarily arise here?

Plane curves. Shouldn't we understand integral / finite type invariants of plane curves, in the style of Arnold's J^+, J^-, and St [Ar], a bit better?


Arnold

St	1	0	0	0	0	1	2	3	· · ·
J^+	0	2	0	0	0	−2	−4	−6	· · ·
J^-	0	0	−2	−1	0	−3	−6	−9	· · ·

References.
[Ar] V. I. Arnold, *Topological Invariants of Plane Curves and Caustics*, University Lecture Series **5**, American Mathematical Society 1994.
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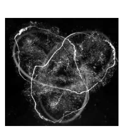
Continuing Joost Slingerland. . .

http://youtu.be/YCA0VIExVhge

http://youtu.be/mHyTOcfF99o



"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

www.katlas.org

A 4DC Discussion.

Definition. A d-dimensional *balanced diagram* is a red-black S&M diagram whose half-edges can be weighted with non-negative integers so that the weight of red edges is $d-1$, the weight of black edges is $d-3$, the weight of M vertices is d and the weight of M vertices is $d-2$.

Question. What are all d-dimensional balanced diagramss?

Safekeeping / Recycling.