The Berezin Integral

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Parallel to http://en.wikipedia.org/wiki/Grassmann_integral

Definition

Linear and satisfies $\int \theta\, d\theta = 1$, $\int 1\, d\theta = 0$ so that

$$\int \frac{\partial}{\partial \theta} f(\theta)\, d\theta = 0$$

Multiple variables.

use "Fubini": $\int f_1(\theta_1) \dots f_n(\theta_n)\, d\theta_1 \dots d\theta_n = \left(\int f_1 d\theta_1\right) \dots \left(\int f_n d\theta_n\right)$

In math talk: V a v.s. $\theta \in V$, $d\theta \in V^*$ w/ $d\theta(\theta) = 1$.

$$f \mapsto \int f\, d\theta \quad \text{is the map} \quad \Lambda(V) \to \Lambda(V)$$

commonly known as "interior multiplication on the left" by $d\theta$:

$$\int f\, d\theta := i_{d\theta}(f)$$

with

$$\int f\, d\theta_1 \dots d\theta_n := f /\!/ i_{d\theta_1} /\!/ i_{d\theta_2} /\!/ \dots /\!/ i_{d\theta_n}$$

$\frac{\partial}{\partial \theta}$ is also $i_{d\theta}$

If $\theta_i = \theta_i(\xi_j)$ and $J_{ij} = \frac{\partial \theta_i}{\partial \xi_j}$ then

$$\int f(\theta_i)\, d\theta = \int f(\theta_i(\xi_j)) \det(J_{ij})^{-1}\, d\xi$$

In math talk: Given V_{θ_i}, $d\theta = \bigwedge d\theta_i \in \Lambda^{top}(V^*)$

W_{ξ_j}, $d\xi = \bigwedge d\xi_j \in \Lambda^{top}(W^*)$ and $T: V_{\theta_i} \to \Lambda^{odd}(W_{\xi_j})$,

T induces a map $T_* : \Lambda V \longrightarrow \Lambda W$, and using it,

$$\int f \, d\theta = \int (T_* f) \cdot \det\left(\frac{\partial (T\theta_i)}{\partial \xi_j}\right)^{-1} d\xi$$

pf. TBW.

Something about changing even & odd variables simultaneously.

Gaussian Integration:

$$\int e^{\theta^T A \eta} \, d\theta \, d\eta = \det A \qquad [\text{for any matrix } A]$$

Proof: Easy.

$$\int e^{\theta^T A \eta + \theta^T J + K^T \eta} \, d\theta \, d\eta = (\det A) \, e^{-K^T A^{-1} J}$$