Chia-Cheng Liu: An introduction to Koszul duality

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I. Example: The classical Koszul Complex. — a projective resolution of k (a field) as a graded $S(k^n)$ module.

II. Quadratic alg. and Koszulity.

III Duality

IV Parabolic – Singular duality.

I k a field, $V = k^n$ basis x_i

$$S(V) := \frac{T_k(V)}{x_i x_j = x_j x_i} \cong k[x_1 \ldots x_n]$$

$$\Lambda(V) := \frac{T_k(V)}{x_i x_j + x_j x_i,\ x_i^2} \cong \mathrm{Span}_k\{x_{i_1} \wedge \ldots \wedge x_{i_m}\}$$

$$\begin{array}{c} \Lambda(V) \\ \cup \\ \Lambda^l(V) \end{array}$$

$$\longrightarrow S(V) \otimes \Lambda^2 \longrightarrow S(V) \otimes \Lambda^1(V) \longrightarrow S(V) \xrightarrow[x_i \mapsto 0]{} k$$

$$p \otimes x_i \longmapsto p \cdot x_i$$

$$p \otimes x_i \wedge x_j \longmapsto (p x_i) \otimes x_j - (p x_j) \otimes x_i$$

In general, . . .

The classical Koszul duality (BGG, 1978)

$$D^b(S(V)\text{-fmod}) \xrightarrow{\sim} D^b(\Lambda(V)\text{-fmod})$$

II k is now a semi-simple ring.

V – a k-k bi-module

$$T_k(V) = k \oplus V \oplus V^{\otimes 2} \oplus \cdots$$

Let $A = \bigoplus A_i = T_k(V)/(R)$ $\quad R \subset V^{\otimes 2}$

"a quadratic ring"

The quadratic dual

$$A^! := T_k(V^*)/(R^\perp) \qquad V^* = \mathrm{Hom}(V, k)$$
$$R^\perp = \{f \in (V^{\otimes 2})^* : f(R) = 0\}$$

Koszul Complex:

$$\longrightarrow A \otimes_k {}^*(A_2^!) \longrightarrow A \otimes_k {}^*(A_1^!) \longrightarrow k$$

where for right k-modules we write

$${}^*W = \mathrm{Hom}_{-k}(W, k)$$

$$d : A \otimes {}^*(A_{i+1}^!) \longrightarrow A \otimes {}^*(A_i^!)$$
$$\|$$
$$f \in \mathrm{Hom}_{-k}(A_{i+1}^!, A) \longmapsto (df)(a) = \sum_\beta f(a v_\beta^\vee) v_\beta$$

<u>Def</u> A positively graded ring A is Koszul if

(1) A_0 is semi-simple $(k := A_0)$

(2) There is a projective resolution of k as a graded A-module

$$\cdots \to P^2 \to P^1 \to P^0 \twoheadrightarrow A_0$$

w/ $P^i = AP^i_i$

Thm 1. Any Koszul ring is quadratic.

2. Let A be a quadratic ring. Then the Koszul complex is a resolution iff A is Koszul.

III Duality