Abstract. I will describe a semi-rigorous reduction to computable combinatorics of perturbative BF theory (Cattaneo-Rossi [CR]), in the case of ribbon 2-links. Also, I will explain how and why my approach may or may not work in the non-ribbon case. Weak this result is, and at least partially already known (Watanabe [Wa]), Yet in the ribbon case, the resulting invariant is a universal finite type invariant, a gadget that significantly generalizes and clarifies the Alexander polynomial and that is closely related to the Kauffman-Vogel problem. I cannot rule out the possibility that the corresponding gadget in the non-ribbon case will be as interesting.

BF Following [CR]. $A \in \Omega^2(M = \mathbb{R}^2, g), B \in \Omega^3(M, g)$, $S(A, B) = \int_M (B, F_0)$. With $f: S^2 \to M, \xi \in \Omega^2(S, \alpha, \beta \in \Omega^2(S, \alpha)$, set

$O(A, B, f) := \int_{S^2} \Omega^2(D) \exp \left( \frac{1}{h} \int_{S^2} \xi \cdot df + f \cdot R \right)$

Deckers Sets ("2D Gauss Codes").

Some Examples.

A 4D knot by Carter and Sato [C-S].

A 4D knot by Dalvit [D].

A 2-link

A Partial Reduction of BF Theory to Combinatorics, 1

The BF Feynman Rules. For an edge $e$, let $\phi_i$ be its direction, in $S^1$ or $S^2$. Let $\alpha_1$ and $\alpha_2$ be volume forms on $\Sigma$ and $\Sigma_1$. Then for a 2-link $\Sigma$, $\gamma_0 \in \Sigma, \gamma_1 \in \Sigma_1$,

$\zeta = \log \sum_{\text{diagrams in } D} \prod_{i=1}^{\infty} \Phi_{\phi_i} \Phi_{\phi_i} \Phi_{\phi_i}$

is an invariant in $CH(FL(T)) \to CH(T)$, "cyclic words in $T$".

A BF Feynman Diagram.
A Partial Reduction of BF Theory to Combinatorics, 2

Theorem 1. For any ribbon 2-knot/link, $c$ can be computed as follows:

$$
\sum_{\text{4-valent}} (+1)^n (-1)^{m-n} k^{m-n} e_n
$$

Theorem 2. Using Gauss diagrams to represent knots and 2-component pure tangles, the above formulas define an invariant in $\text{CW}(\mathbb{F}_2 (\mathcal{C})) \rightarrow \text{CW} (\mathcal{C})$, “cyclic words in $\mathcal{C}$”.

- Agrees with BN-Dancso (BND) and with |BN1|.
- In-practice computable:
  - Vanishes on braids.
  - Extends to $\mathcal{C}$.
- Contains Alexander.

The “missing factor” in Levine’s factorization [Le] (the rest of [Le]) also fits, hence contains the MVA.

Related to $\varepsilon$ extends Farber’s [Far]?

Should be summed and categorified.

Sketch of Proof. In 4D space, for a “drop down” red propagator, hence no $M$-trivalent vertices. $s$-integrals are 1 if “ground pieces” run on nested curves as below, and exponentials arise when several propagators compete for the same double curve. And then the combinatorics is obvious...

Musings

Chern-Simons. When the domain of BF is restricted to ribbon knots, and the target of CS is restricted to trees and wheels, they agree. Why?

Is this all? What about the $\varepsilon$-invariant? (the “true” triple linking number)

Knots. In 3D, a generic immersion of $S^1$ is an embedding, a knot. In 4D, a generic immersion of a surface has finitely-many double points (a grain?) Perhaps we should be studying these?

Finite type. What are finite-type invariants for 2-knots? What would be “chord diagrams”?

Hubble-wrap-finite-type

There’s an alternative definition of finite type in 3D, due to Goussarov (see [BND]). The obvious parallel in 4D involves “bubble wraps”. Is it any good?

Shielded tangles. In 3D, one can’t zoom in and compute “the Chern-Simons invariant of a tangle”. Yet there are well-defined invariants of “shielded tangles”, and rules for their compositions. What would the 4D analog be?

Will the relationship with the Khishvara-Vergne problem [BND] necessarily arise here?

Plane curves. Shouldn’t we understand integral / finite type invariants of plane curves, in the style of Arnold’s $J^*$, $J^+$, and $s^r$ [Ar], a bit better?

Are v's connected to the knots, all else in topology is the work of mortals.

virtual 2-knots? 6-group only?

References


2014-02 Page 2