Integrable systems .... KdV eqn’s, waves in shallow water:

\[ U_t = U U_{xx} + U_{xxxx} \]

Non-linear Schrödinger (NLS)

\[ i \psi_t = \psi_{xx} + 1/4 |\psi|^2 \psi \]

Sine-Gordon eqn

\[ U_t - U_{xx} = \sin U \]

Toda lattice

\[ \ddot{x}_i = e^{x_i-x_{i+1}} - e^{x_{i-1}-x_i} \]

KP

(waves in plasma)

\[ \sigma^2 U_{yy} = (U_t - U U_{xx} - U_{xxxx})_x \]

What is in common?

\{integrable systems\} \iff \{a compatibility condition of an over-determined system of linear eqns\}

Example

\[ L \Psi = E \Psi \]

\[ (\partial^2_t - A) \Psi = 0 \]

Compatibility:

\[ [L, \partial^2_t - A] = 0 \iff L' = [A, L] \Rightarrow \text{ "Lax Form" } \]

\[ L = \left( \partial^2_{xx} + U(x,t) \right) \quad A = \left( \partial^3_{xx} + \frac{3}{2} U U_{xx} + \frac{3}{4} U_{x} \right) \]
Compatibility becomes KdV, up to minor normalizations.

Why is it good?

Get conserved quantities:

\[(tr L^k)' = 0\]