The Kashiwara-Vergne Problem and Topology

Abstract. I will describe a general machine, a close cousin of Taylor's theorem, whose inputs are topics in topology and whose outputs are problems in algebra. There are many inputs the machine can take, and many outputs it produces, but I will concentrate on just one input/output pair. When fed with a certain class of knotted 2-dimensional objects in 4-dimensional space, it outputs the Kashiwara-Vergne Problem (1978 ω/KV, solved Alekseev-Meinrenken 2006 ω/AM, elucidated Alekseev-Torossian 2008-2012 ω/AT), a problem about convolutions on Lie groups and Lie algebras.

The Kashiwara-Vergne Conjecture.

There exist two series F and G in the completed free Lie algebra in generators x and y so that

$$x+y-\log e^y e^x = (1-e^{-\operatorname{ad} x})F + (e^{\operatorname{ad} y}-1)G$$

and so that with $z=\log e^x e^y$,

$$\operatorname{tr}(\operatorname{ad} x)\partial_x F + \operatorname{tr}(\operatorname{ad} y)\partial_y G = \frac{1}{2}\operatorname{tr}\left(\frac{\operatorname{ad} x}{e^{\operatorname{ad} x}-1}+\frac{\operatorname{ad} y}{e^{\operatorname{ad} y}-1}-\frac{\operatorname{ad} z}{e^{\operatorname{ad} z}-1}-1\right)$$

Implies the loosely-stated convolutions statement: Convolutions of invariant functions on a Lie group agree with convolutions of invariant functions on its Lie algebra.


Kashiwara


Vergne


Alekseev


Meinrenken


Torossian

The Machine.

Let G be a group, $\mathcal{K}=\mathbb{Q}G=\{\sum a_i g_i : a_i\in\mathbb{Q}, g_i\in G\}$ its group-ring, $\mathcal{I}=\{\sum a_i g_i : \sum a_i = 0\}\subset\mathcal{K}$ its augmentation ideal. Let

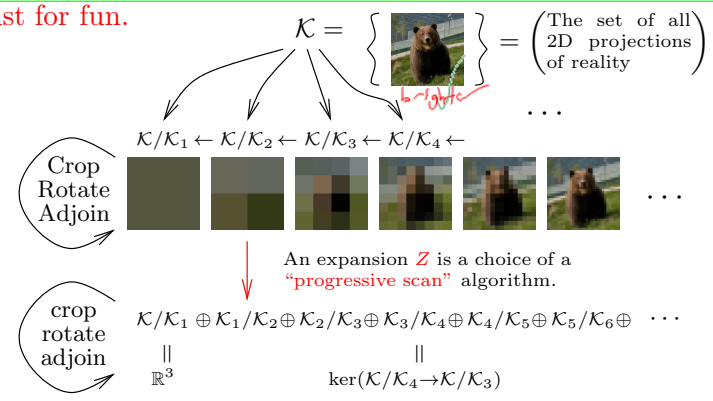
$$\mathcal{A}=\operatorname{gr}\mathcal{K}:=\widehat{\bigoplus_{m\ge 0}}\mathcal{I}^m/\mathcal{I}^{m+1}.$$

Note that $\mathcal{A}$ inherits a product from G.

Definition. A linear $Z:\mathcal{K}\to\mathcal{A}$ is an "expansion" if for any $\gamma\in\mathcal{I}^m$, $Z(\gamma)=(0,\dots,0,\gamma/\mathcal{I}^{m+1},*,\dots)$, and a "homomorphic expansion" if in addition it preserves the product.

Example. Let $\mathcal{K}=C^\infty(\mathbb{R}^n)$ and $\mathcal{I}=\{f: f(0)=0\}$. Then $\mathcal{I}^m=\{f: f \text{ vanishes like } |x|^m\}$ so $\mathcal{I}^m/\mathcal{I}^{m+1}$ degree m homogeneous polynomials and $\mathcal{A}$ is power series. The Taylor series is a homomorphic expansion!

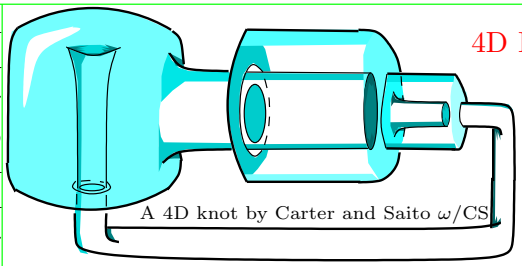
Just for fun.

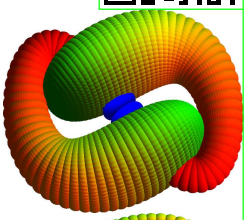


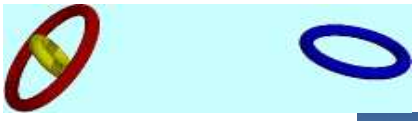
Theorem (with Zsuzsanna Dancso, ω/WKO). Finding a homomorphic expansion for $w\mathcal{K}$ is equivalent to solving the Kashiwara-Vergne problem. There is a bijection between the set of homomorphic expansions for $w\mathcal{K}$ and the set of solutions of the Kashiwara-Vergne problem.

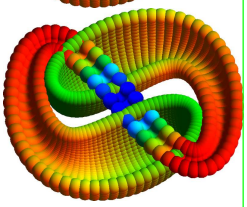

Dancso

4D Knots.





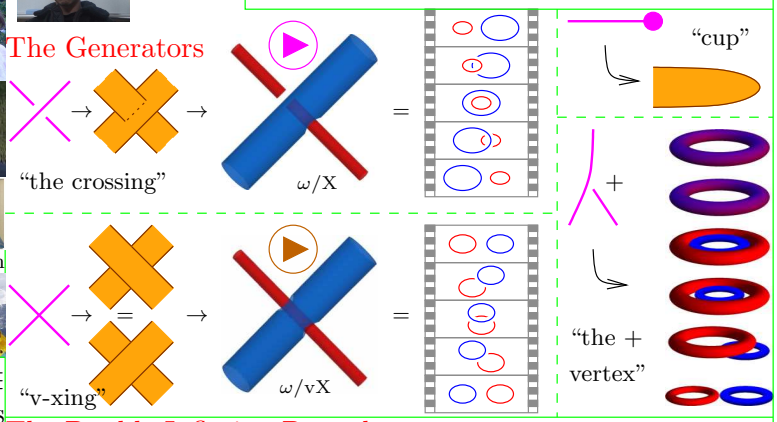

ω/F



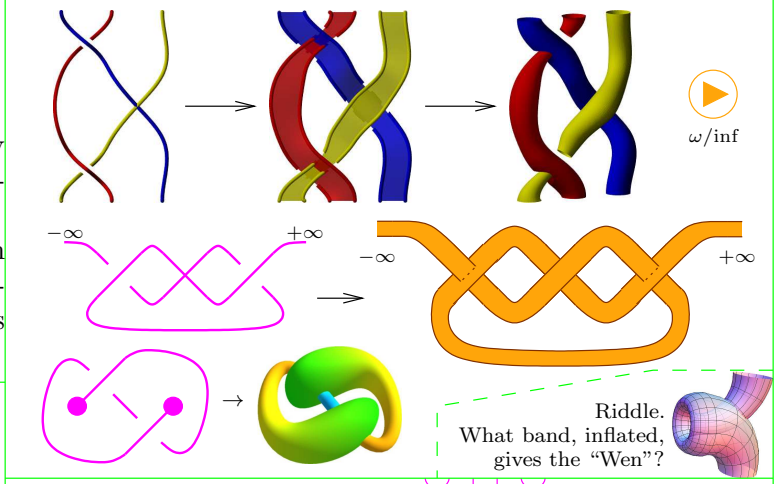

Satoh

Dalvit
ω/Dal

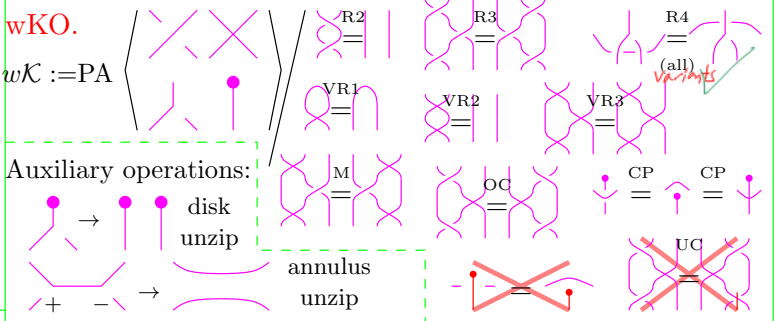
The Generators



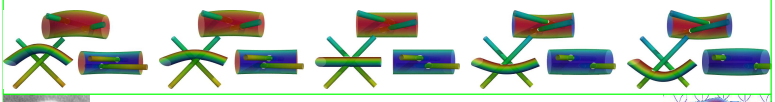
The Double Inflation Procedure.



wKO.



Proof of Relation M:



"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

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