Uniqueness of Taylor

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3:46 AM

$R = C^\infty(\mathbb{R})$, $I = \{f : f(0) = 0\}$, $A = \bigoplus I^m / I^{m+1}$

$Z : R \to A$ s.t. if $f \in I^m$,

$$Z(f) = (0 \dots 0, f/I^{m+1}, *, * \dots)$$

$$Z(f) = (f(0), xf'(0) + axf(0), \tfrac{x^2}{2} f''(0) + \dots, \dots)$$

So $Z(f) = \sum_{m=0}^{\infty} \frac{1}{m!} P_m(x) \frac{\partial^m}{\partial x^m} f$ where $P_m(x) = x^m +$ higher terms

When is $Z(fg) = Z(f)Z(g)$?
"homomorphic"

Guess: Exactly when $Z(f) = Z_T(f \circ \Phi)$,
where Z_T is the ordinary Taylor expansion
and $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ has $\Phi(0) = 0$ & $d\Phi_0 = Id$.

When is $(Z_x \otimes Z_y) f(x+y) = (Zf)(x+y)$?
"co-homomorphic"

$x^m \mapsto P_m(x) \quad y^m \longmapsto P_m(y)$

$x + y \longmapsto P_1(x) + P_1(y) \overset{?}{=} P_1(x+y)$

$f = z^2$:

$(Z_x \otimes Z_y) f(x+y) = (Z_x \otimes Z_y)(x^2 + 2xy + y^2)$

$$= P_2(x) + 2xy + P_2(y)$$

$$\overset{?}{=} P_2(x+y)$$

$$P_2(z) = z^2 + h(z)$$

$$h(x) + h(y) \overset{?}{=} h(x+y)$$

$$\Rightarrow h = 0.$$

Guess: The only co-homomorphic z is z_T.