"We don't want to be over-swiss, so we wait one minute."

Kähler groups: $\Gamma$ is a Kähler group if it is $\pi_1$ of a closed Kähler manifold. A Kähler group is finitely presented.

There is no characterization in sight.

**Def** $\Gamma$ is a projective group if it is $\pi_1$ of a smooth projective variety $X$.

$$\mathcal{P} \subseteq \mathcal{K} \subseteq \mathcal{G}$$

- projective groups
- Kähler groups
- finitely presented

Examples 0. Every finite group is Kähler, in fact projective.

1. A finitely generated Abelian group is Kähler/projective if its rank is even.

2. Free groups are not Kähler.
### Negative results
- **Thm. (Gromov):** Kähler groups have finitely many ends.
- **Thm. (Carlson-Toledo):** If $\Gamma'$ is a lattice (co-compact) in $\text{Is}(\mathbb{H}^n), n \geq 3$, then $\Gamma'$ is not Kähler.
- **Thm**. 1989: The Higman 4-group is not Kähler.

### Positive results
- **Toledo (1990):** $\exists$ Kähler group of every cohomological dimension $\geq 5$.
- **Toledo (1993):** $\exists$ non-residually finite Kähler groups (in fact projective).
- **M. Kojima 1998:** $\exists$ non-coherent Kähler groups $\text{[have finitely generated subgroups that are not finitely presented]}$

#### Thm 1.
Thompson's group $F$ is not Kähler
(Napier-Ramachandra, 2005)

#### Thm 2.
A group of deficiency 2 is Kähler iff it is $\pi_1\text{orb}(C_g)$

**Deficiency:** $\max_{\text{all}} \left( \# \text{gens} - \# \text{rels} \right)$
Thm 3. An infinite 3-manifold group is Kähler iff it is $\pi_1(Cg)$.

The argument for all 3 theorems:

$X$: compact Kähler manifold, connected

$\text{Alb}_x : X \rightarrow H^0(X,\Omega^1_X) / H^1(X,\mathbb{Z})$

"The Albanese map" by integrating paths from a fixed basepoint.

$C : X \rightarrow C$ is a holomorphic surjective map with compact fibers to a curve $C$.

Then

$\begin{array}{ccc}
1 & \rightarrow & K & \rightarrow \pi_1(X) & \rightarrow \pi_1(C) & \rightarrow 1 \\
& & & \text{3or4mins.} & & \\
\end{array}$