Abstract. Much as we can understand 3-dimensional objects by staring at their pictures and x-ray images and slices in 2-dimensions, so can we understand 4-dimensional objects by staring at their pictures and x-ray images and slices in 3-dimensions, capitalizing on the fact that we understand 3-dimensions pretty well. So we will spend some time staring at and understanding various 2-dimensional views of a 3-dimensional elephant, and then even more simply, various 2-dimensional views of some 3-dimensional knots. This achieved, we’ll take the leap and visualize some 4-dimensional knots by their various traces in 3-dimensional space, and this achieved, I will tell you about the simplest problem in 4-dimensional knot theory whose solution I don’t know.

Visualizing the Fourth Dimension

A 4D knot by Carter and Saito

with Ester Dalvit

with Jeff2207

with Carter-Saito

The Double Inflation Procedure \( \delta \).

\[ \delta : \quad \cdot \cdot \cdot \quad \cdot \cdot \cdot \quad \cdot \cdot \cdot \quad \cdot \cdot \cdot \quad \cdot \cdot \cdot \quad \cdot \cdot \cdot \]

\[ \omega /\text{M}2 \]

\[ \omega /\text{m}1 \]

“long w-knot diagram”

“long knotted 2D tube in 4D”

Many of the images are by Carter and Carter-Saito.
God created the knots, all else in topology is the work of mortals.

Leopold Kronecker (modified)

Satoh’s Conjecture. The “kernel” of the “double inflation” map $\delta$, mapping “long” $w$-knot diagrams in the plane to “long” knotted 2D tubes in 4D, is precisely the moves $R1–R3$, $VR1–VR3$, $D$ and $OC$ listed below.

In other words, two long $w$-knot diagrams represent via $\delta$ the same long 2D knotted tube in 4D iff they differ by a sequence of the said moves.

First Iso. Thm: $\phi: G \rightarrow H \Rightarrow \text{im}\, \phi \cong G/\ker(\phi)$

$\delta$ is a map from algebra to topology. So a thing in “hard” topology (“ribbon 2-knots”) is the same as a thing in “easy” algebra.

What’s “The Same”?

Reidemeister’ Theorem. Two knot diagrams represent the same 3D knot iff they differ by a sequence of “Reidemester moves”:

- $R1$
- $R2$
- $R3$

Some add:

- $D$
- $OC$

w-Moves. Same $R1$, $R2$, $R3$ as above, and also:

- $VR1$
- $VR2$
- $VR3$

And the Simplest Thing I Don’t Know About It

Movie Moves.

Roseman Moves.

Surfaces and Movies.