A Quick Introduction to Khovanov Homology

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Why Bother?

Khovanov: $K(L)$ is a chain complex of graded $\mathbb{Z}$-modules:

$V = \text{span}(e_+, e_-); \quad \deg e_+ = 1; \quad \deg e_- = 1,$

$K(\bigwedge^k V) = V^k; \quad K(\bigvee^{n+k} V) = \bigoplus_{\pi \in S_n} \text{span}(\pi(e_1, \ldots, e_n) \otimes e_{-1}^{d_{\pi 1}} \otimes \cdots \otimes e_{-n}^{d_{\pi n}}), \quad \deg e_+ = 1, \quad \deg e_- = 0,$

$K(\bigwedge^k V) = \bigoplus_{\pi \in S_n} \text{span}(\pi(e_1, \ldots, e_n) \otimes e_{-1}^{d_{\pi 1}} \otimes \cdots \otimes e_{-n}^{d_{\pi n}}), \quad \deg e_+ = 1, \quad \deg e_- = 0,$

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Example:

$q^2(q + q^{-1})^2 = 3q^2(q + q^{-1}) + 3q^2(q + q^{-1})^2 - q^4(q + q^{-1})^3.$

Story/Theorem. To every finite-dimensional metrized Lie algebra $g$ and a list $R_1, \ldots, R_n$ of representations thereof, there is an associated invariant of re-arranging links, valued in Laurent polynomials in a variable $q$ (really, in characters of the space of metrics on $g$).

Queries. Do you know how to define/prove this? Really? With all the details? Could you teach the story/proof leaving no black boxes in a one-semester course on knot theory to students who are not already experts in Lie algebras? Do I really need to know about Cartan subalgebras and root and weight spaces? Is this theorem at all true?


The Jones polynomial:
\[ J : \mathcal{K} \mapsto q^{\text{link length}} \cdot (q + q^{-1})^{|\mathcal{K}|} \]

R2

Local Khovanov Homology (1)

(an outdated overview)

What is it?
A cube for each knot/link projection;
Vertices: All fillings of
Edges: All fillings of \( I \times \mathcal{K} \)
with \( I \times \)

Signs?

More crossings?

Where does it live?
In \( \text{Kom}(\text{Mat}(\text{Cob}) / (S, T, G, NC)) / \text{homotopy} \)
Knots: Complexes \( \text{Mat} \): Matrices
Cob: Cobordisms \( <\ldots> \): Formal lin. comb.

Computable!
via "complex simplification"

Complexes:
\[ \Omega = \{ \Omega^n \} \quad \text{such that} \quad \Omega^n = \text{the category of} \quad \Omega^{n+1} \]
Morphisms:
\[ F : (\ldots \text{to} \quad \ldots) \]
Homotopies:
\[ G : (\ldots \text{to} \quad \ldots) \]

The Main Point. The cube, \( \text{Kh}(L) \), is an up-to-homotopy invariant of knots and links. It’s Euler characteristic is the Jones polynomial, yet it is strictly stronger than the Jones polynomial. It is functorial (in the appropriate sense) and practically computable.

The Categorification Speculative Paradigm.
- Every object in math is the Euler characteristic of a complex.
- Every operation lifts to an operation between complexes.
- Every identity remains true, up to homotopy.
Local Khovanov Homology (2)

The case of tangles:

The Reduction Lemma. If $\phi$ is an isomorphism then the complex

$$[C] \xrightarrow{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}} \begin{pmatrix} b_0 \\ D \end{pmatrix} \xrightarrow{\begin{pmatrix} \phi & \gamma \\ 0 & \delta - \gamma \phi^{-1} \delta \end{pmatrix}} \begin{pmatrix} b_2 \\ E \end{pmatrix} \xrightarrow{\begin{pmatrix} \mu & \nu \end{pmatrix}} [F]$$

is isomorphic to the (direct sum) complex

$$[C] \xrightarrow{\begin{pmatrix} 0 & \beta \\ \gamma & \delta \end{pmatrix}} \begin{pmatrix} b_0 \\ D \end{pmatrix} \xrightarrow{\begin{pmatrix} \phi & \gamma \\ 0 & \delta - \gamma \phi^{-1} \delta \end{pmatrix}} \begin{pmatrix} b_2 \\ E \end{pmatrix} \xrightarrow{\begin{pmatrix} 0 & \nu \end{pmatrix}} [F]$$

Invariance under R2.

After delooping:

I mean business. $T(7,6)$ says

In 1 day $\dim H_r$ is given by:

| $r$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |
| 57  | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 58  | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 59  | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 60  | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

Functoriality / cobordisms.

J. Rasmussen: Leads to a no-analysis proof of a conjecture by Milnor.

A more general theory: Remove $G$ and $NC$, add

$4Tu$: Remove $G$ and $NC$, add

(minor further revisions are necessary)

Visit!

Edit!

"God created the knots, all else is topology is the work of man".

Leopold Kronecker (modified)

http://katlas.org