

**A Quick Introduction to Khovanov Homology**

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**Abstract.** I will tell the Kauffman bracket story of the Jones polynomial as Kauffman told it in 1987, then the Khovanov homology story as Khovanov told it in 1999, and finally the "local Khovanov homology" story as I understood it in 2003. At the end of our 90 minutes we will understand what is a "Jones homology", how to generalize it to tangles and to cobordisms between tangles, and why it is computable relatively efficiently. But we will say nothing about more modern stuff – the Rasmussen invariant, Alexander and HOMFLYPT knot homologies, and the categorification of  $sl_2$  and other Lie algebras.

**Why Bother?**



What is Categorification=Concretization=de-abstractation? "3" is {cow, cow, cow} and {pig, pig, pig} and many other things...

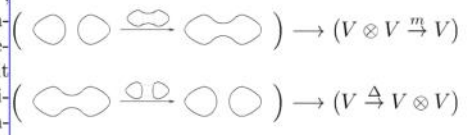
Natural numbers  $\mapsto$  finite sets, equalities  $\mapsto$  bijections, inequalities  $\mapsto$  injections and surjections:  

$$\binom{2n}{n} = \sum \binom{n}{k} \mapsto \binom{X \times \{1,2\}}{|X|} \leftrightarrow \bigcup \binom{X}{k} \times \binom{X}{k}$$

**Khovanov:**  $K(L)$  is a chain complex of graded  $\mathbb{Z}$ -modules;  
 $V = \text{span}(v_+, v_-)$ ;  $\text{deg } v_{\pm} = \pm 1$ ;  $q\text{dim } V = q + q^{-1}$ ;

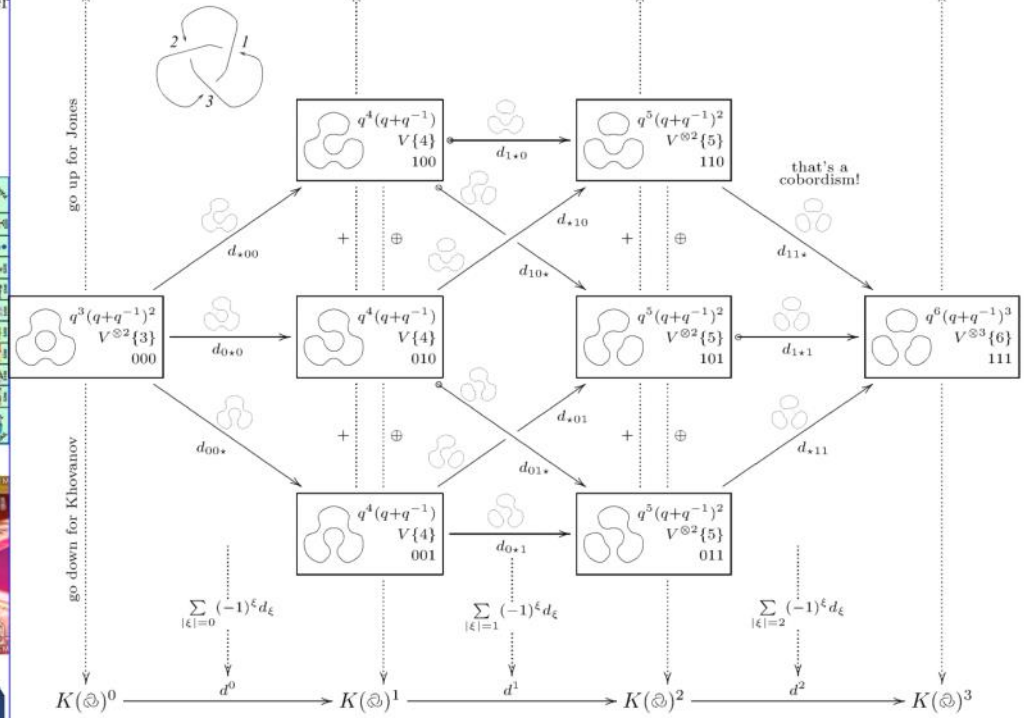
$$K(\circlearrowleft^k) = V^{\otimes k}; \quad K(\times) = \text{Flatten} \left( 0 \rightarrow K(\circlearrowleft^1)\{1\} \rightarrow K(\circlearrowright^2)\{2\} \rightarrow 0 \right);$$

$$K(\times) = \text{Flatten} \left( 0 \rightarrow K(\circlearrowright^2)\{-2\} \rightarrow K(\circlearrowleft^1)\{-1\} \rightarrow 0 \right);$$



Example:

$$q^3(q+q^{-1})^2 - 3q^4(q+q^{-1}) + 3q^5(q+q^{-1})^2 - q^6(q+q^{-1})^3$$

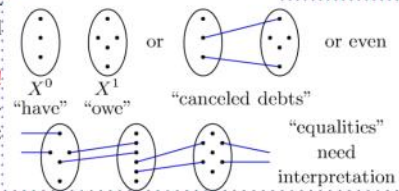


(here  $(-1)^\xi := (-1)^{\sum_{i < j} \xi_i}$  if  $\xi_j = \star$ ) =  $K(\otimes)$ .

- Theorem 1.** The graded Euler characteristic of  $K(L)$  is  $J(L)$ .
- Theorem 2.** The homology  $\text{Kh}(L)$  of  $K(L)$  is a link invariant.
- Theorem 3.**  $\text{Kh}(L)$  is strictly stronger than  $J(L)$ :  $J(\bar{5}_1) = J(10_{132})$  yet  $\text{Kh}(\bar{5}_1) \neq \text{Kh}(10_{132})$ .

**References.** Khovanov's arXiv:math.QA/9908171 and arXiv:math.QA/0103190 and my <http://www.math.toronto.edu/~drorbn/papers/Categorification/>.

$\mathbb{Z}$ . Negative numbers:



**The Philosophy Corner**



**Weaker Categorification.** Do the same in the category of vector spaces: "3" becomes  $V$  s.t.  $\text{dim } V = 3$ , or better,  $V^\bullet = (\dots V^{r-1} \rightarrow V^r \rightarrow V^{r+1} \dots)$  s.t.  $d^2 = 0$  and  $\chi(V^\bullet) := \sum (-1)^r \text{dim } V^r = 3 = \sum (-1)^r \text{dim } H^r$ . Equalities become homotopies between complexes.

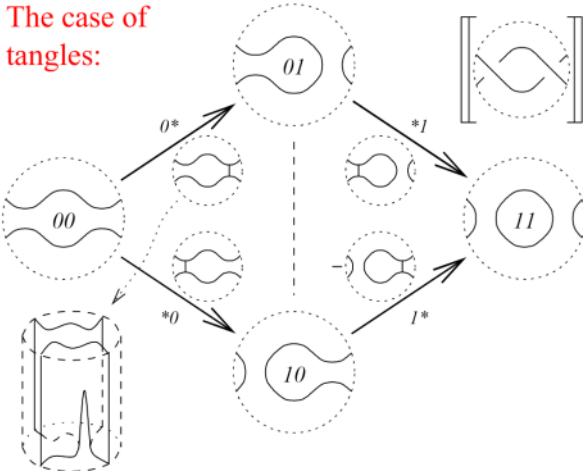
**Categorifying  $\mathbb{Z}[q^{\pm 1}]$ .**  $f = \sum a_j q^j$  becomes  $V = \bigoplus V_j$  s.t.  $q\text{dim } V := \sum q^j \text{dim } V_j = f$ , or better,  $V^\bullet = (\dots V^{r-1} \rightarrow V^r \rightarrow V^{r+1} \dots)$  s.t.  $d^2 = 0$ ,  $\text{deg } d = 0$ , and  $\chi_q(V^\bullet) := \sum (-1)^r q\text{dim } V^r = f = \sum (-1)^r q\text{dim } H^r$ .

**Note.** Setting  $V\{l\}_j := V_{j-l}$ , we get  $q\text{dim } V\{l\} = q^l q\text{dim } V$ .

$$= q + q^3 + q^5 - q^9.$$

## Local Khovanov Homology (2)

The case of tangles:



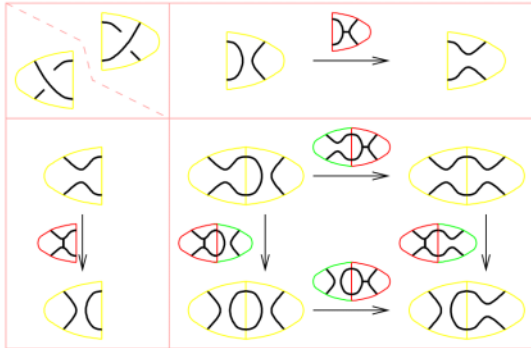
The Reduction Lemma. If  $\phi$  is an isomorphism then the complex

$$[C] \xrightarrow{\begin{pmatrix} \alpha \\ \beta \end{pmatrix}} \begin{bmatrix} b_1 \\ D \end{bmatrix} \xrightarrow{\begin{pmatrix} \phi & \delta \\ \gamma & \epsilon \end{pmatrix}} \begin{bmatrix} b_2 \\ E \end{bmatrix} \xrightarrow{(\mu \ \nu)} [F]$$

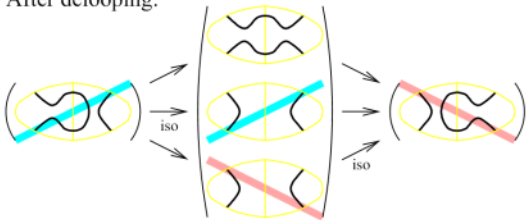
is isomorphic to the (direct sum) complex

$$[C] \xrightarrow{\begin{pmatrix} 0 \\ \beta \end{pmatrix}} \begin{bmatrix} b_1 \\ D \end{bmatrix} \xrightarrow{\begin{pmatrix} \phi & 0 \\ 0 & \epsilon - \gamma\phi^{-1}\delta \end{pmatrix}} \begin{bmatrix} b_2 \\ E \end{bmatrix} \xrightarrow{(0 \ \nu)} [F]$$

Invariance under R2.



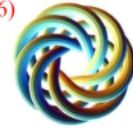
After delooping:



Kurt Reidemeister

I mean business.

T(7,6)



Old techniques:

~1,000 years,  
~1Ggb RAM.

(now down to seconds)

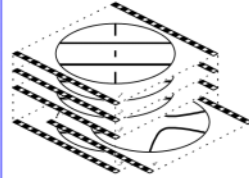


In 1 day says

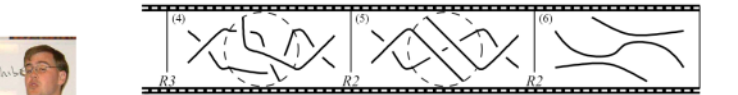
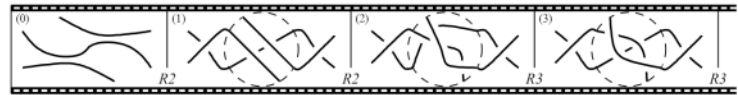
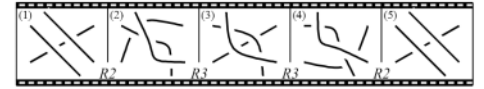
$\dim_j H_r$  is given by:

$j \setminus r$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	
57																					
55																					1
53																					1 1
51																					1 2
49																					1 1 2
47																					3 1 1
45																					2 1 1
43																					3 1 1
41																					1 2 1
39																					1 1 2
37																					1 1 1
35																					1 1 1
33																					1 1 1
31																					1 1 1
29																					1 1 1

Functoriality / cobordisms.



M. Jacobsson



J. Rasmussen: Leads to a no-analysis proof of a conjecture by Milnor.

A more general theory: Remove G and NC, add

$$4Tu: \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} + \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} = \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} + \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

(minor further revisions are necessary)

"God created the knots,  
all else in topology is the work of mortals"

Leopold Kronecker (modified)



Visit!

Edit!

<http://katlas.org>

<http://www.math.toronto.edu/~drorbn/papers/Cobordism/>  
<http://www.math.toronto.edu/~drorbn/papers/FastKh/>  
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