Meta–Groups, Meta–Bicrossed–Products, and the Alexander Polynomial, 1

Dror Bar–Natan in Montreal, June 2013

http://www.math.toronto.edu/~drorbn/fails/Montreal-1947/}

Abstract. I will define “meta-groups” and explain how one specific meta-group, which in itself is a “meta-bicrossed-product”, gives rise to an “ultimate Alexander invariant” of tangles, that contains the Alexander polynomial (multivariable, if you wish), has extremely good composition properties, is evaluated in a topologically meaningful way, and is least-wasteful in a computational sense. If you believe in categorification, that’s a wonderful playground. This work is closely related to work by Le Dinet (Comment. Math. Helv. 67 (1992) 306-315), Kirk, Livingston and Wang (arXiv:math/9809035) and Cimasoni and Turaev (arXiv:math.GT/0406269).


Alexander Issues.
- Quick to compute, but computation departs from topology.
- Extends to tangles, but at an exponential cost.
- Hard to categorify.

Idea. Given a group $G$ and two “$YB$” pairs $R^2 = (g_0^F, g_1^F) \in G^2$, map them to xings and “multiply along”, so that

$$\begin{align*}
\frac{\pm}{Z} & \rightarrow \frac{\pm}{Z} \\
go^+_0 & \rightarrow \frac{\pm}{Z} \\
go^+_1 & \rightarrow \frac{\pm}{Z}
\end{align*}$$

This False! R2 implies that $g_0^2 g_1^2 = e = g_0^2 g_2^2$ and then R3 implies that $g_0^e$ and $g_1^e$ commute, so the result is a simple counting invariant.

A Group Computer. Given $G$, can store group elements and perform operations on them:

$$x : g_1 \quad u : g_2 \quad y : g_4 \quad z : g_4$$

Also has $S_c$ for inversion, $c_e$ for unit insertion, $d_i$ for register deletion, $\Delta_{ij}$ for element cloning, $p^i$ for remapping, and $(D_1, D_2) \rightarrow D_1 \cup D_2$ for merging, and many obvious composition axioms relating these.

$$P = \{ x : g_1, y : g_2 \} \Rightarrow \bar{P} = \{ d_i P \} \cup \{ d_i P \}$$

A Meta–Group. Is a similar “computer”, only its internal structure is unknown to us. Namely it is a collection of sets $\{G_\gamma\}$ indexed by all finite sets $\gamma$, and a collection of operations $m^{\gamma}_{\delta_{ij}}$, $S_c$, $c_e$, $d_i$, $\Delta_{ij}$ (sometimes), $p^i$, and $\cup$, satisfying the exact same linear properties.

Example 0. The non–meta example, $G_\gamma : = G_\gamma^1$.

Example 1. $G_\gamma : = M_{n \times n}(Z)$, with simultaneous row and column operations, and “block diagonal” merges. Here if $P = \{ x : a b \}$ then $d_i P = \{ x : a \}$ and $d_i P = \{ y : d \}$ so

$$\{ d_i P \} \cup \{ d_i P \} = \{ x : a 0 \} \neq P.$$ 

So this $G$ is truly meta.

Claim. From a meta-group $G$ and $YB$ elements $R^2 \in G_2$ we can construct a knot/tangle invariant.

Bicrossed Products. If $G = HT$ is a group presented as a product of two of its subgroups, with $H \cap T = \{ e \}$, then also $G = TH$ and $G$ is determined by $H$, $T$, and the “swap” map $sw^{ht} : (t, h) \mapsto (h', t')$ defined by $th = h't'$. The map $sw^{ht}$ satisfies (1) and (2) below; conversely, if $sw : T \times H \mapsto H \times T$ satisfies (1) and (2) (++ lesser conditions), then (3) defines a group structure on $H \times T$, the “bicrossed product”.

(1) $sw^{ht}(t, h) = (h, t)$

(2) $sw^{ht}(t_1, h_1) sw^{ht}(t_2, h_2) = sw^{ht}(t_1 h_2 t_2^{-1} h_2^{-1}, t_1 t_2)$

(3) $sw^{ht} : (t, h) \mapsto (h', t')$ defined by $th = h't'$.
A: People study $\mathcal{T}_f(X) = [\mathcal{O}_b, X]$ & $\mathcal{T}_c(X) = [\mathcal{O}_b, X]$ why not $\mathcal{T}_f(X) = [\mathcal{O}_b^b, X]$?