“Knot homology and the index”

\[ J_k(M) \leftrightarrow \text{The Jones polynomial, comes from } SU(2) \text{ CS on } S^3 \text{ w/ Wilson loop} \]

\[ Z(M) = \int DA \ e^{i k \mathcal{S}_{CS}(M)} \quad \text{no metric} \]

\[ \mathcal{S}_{CS} = \int_M \mathrm{An} \mathrm{d} A + \frac{2\pi}{3} \mathrm{An} \mathrm{d} A \wedge A \wedge A \quad k = \frac{1}{h} \]

... topological invariants of M.

\[ k \rightarrow \operatorname{Tr}_k \ P e^{i k} \Rightarrow \theta_k(k) e^k \]

\[ J_k(q) = \langle \theta_k(k) \rangle_{CS} \quad q = e^{2\pi i / k+2} \]

Why do it? Why is it important?

\( \text{Over: It isn't.} \)

\( \Rightarrow \) makes invariance manifest. [fit invariants]

\( \Rightarrow \) “The Jones poly is a quantum object”

\( \Rightarrow \) “Chern-Simons theory provides the logic behind the zoo of invariants”.

Weak reasons!

\( \Rightarrow \) CS arises elsewhere, and connections
Chern-Simons Theory is Soluble

CS invariants of any 3-manifold \([\mathfrak{g}, k]\) are computable explicitly in terms of the path integral on a manifold \(M\) with \(\partial M = B\) determines a state

\[ Z(M) \in \mathcal{H}_B \quad \text{"The Hilbert space of } B \text{"} \]

In 3D TQFT, \(B = T^2\)

\[ \mathcal{H}_B = \mathcal{H}_{T^2} \text{ is spanned by } \]

\[ Z \left( \begin{array}{c} \begin{array}{c} \infty \end{array} \end{array} \right) = \left| R_1 \right> \]

\[ S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

\[ S^2 = -1, \quad (ST)^3 = S^2 \]

\[ \text{SL}(2, \mathbb{Z}) \text{ acts on } \mathcal{H}_{T^2} \]

\[ \text{Diff} \circ \@ T^2 \quad S \leftrightarrow \text{invariants of Hopf links} \]

\[ B: \text{The braiding matrix.} \]
Every 3-manifold is a surgery on a link.

Write an arbitrary link as a braid closure, use $B$.

What CS does not explain:
One always gets polynomials w/ integer coefficients.

Khoivanov $q$:

\[ K \mapsto H^{i,j}(K) \text{ vs. w/ 2 gradings} \]

\[ S.t. \quad \gamma_k(q) = \sum_{i,j} \frac{(-1)^j}{j!} q^j \dim(H^{i,j}(K)) \]