Cheat Sheet β

http://drorbn.net/AcademicPensieve/2013-05/
initiated 24/3/13; continues 2013-04; modified 9/5/13, 12:17 Noon

The original β-calculus: With $\epsilon := 1+\alpha$, $\langle\alpha\rangle := \sum_v \alpha_v$, and $\langle\gamma\rangle := \sum_{v\neq u}\gamma_v$,

$$\begin{array}{c|c} \omega_1 & H_1 \\ \hline T_1 & A_1 \end{array} * \begin{array}{c|c} \omega_2 & H_2 \\ \hline T_2 & A_2 \end{array} \underset{\beta}{=} \begin{array}{c|cc} \omega_1\omega_2 & H_1 & H_2 \\ \hline T_1 & A_1 & 0 \\ T_2 & 0 & A_2 \end{array} \qquad \begin{array}{c|c} \omega & H \\ \hline u & \alpha \\ v & \beta \\ T & \gamma \end{array} \xrightarrow[\beta]{tm^{uv}_w} \begin{array}{c|c} \omega & H \\ \hline w & (\alpha+\beta) /\!\!/ \left(\begin{smallmatrix} u,v \\ \to w \end{smallmatrix}\right) \\ T & \gamma \end{array} \qquad R^{\pm}_{ux} \underset{\beta}{=} \begin{array}{c|c} 1 & x \\ \hline u & t_u^{\pm 1}-1 \end{array}$$

$$\begin{array}{c|ccc} \omega & x & y & H \\ \hline T & \alpha & \beta & \gamma \end{array} \xrightarrow[\beta]{hm^{xy}_z} \begin{array}{c|cc} \omega & z & H \\ \hline T & \alpha+\beta+\langle\alpha\rangle\beta & \gamma \end{array} \qquad \begin{array}{c|cc} \omega & x & H \\ \hline u & \alpha & \beta \\ T & \gamma & \delta \end{array} \xrightarrow[\beta]{sw^{ux}_{th}} \begin{array}{c|cc} \omega\epsilon & x & H \\ \hline u & \alpha(1+\langle\gamma\rangle/\epsilon) & \beta(1+\langle\gamma\rangle/\epsilon) \\ T & \gamma/\epsilon & \delta-\gamma\beta/\epsilon \end{array}$$

Constraints. • Column sums are monomials minus 1.

β-better calculus: Constraints. • Sum of column x is $(\sigma_x-1)w$. • $\omega^{k-1} \mid \Lambda^k A$. • At $t_*=1$, $\omega=1$ and $A=0$.

$$\begin{array}{c|c} \omega_1 & H_1 \\ - & \sigma_1 \\ \hline T_1 & A_1 \end{array} * \begin{array}{c|c} \omega_2 & H_2 \\ - & \sigma_2 \\ \hline T_2 & A_2 \end{array} \underset{\beta_b}{=} \begin{array}{c|cc} \omega_1\omega_2 & H_1 & H_2 \\ - & \sigma_1 & \sigma_2 \\ \hline T_1 & \omega_2 A_1 & 0 \\ T_2 & 0 & \omega_1 A_2 \end{array} \qquad \begin{array}{c|c} \omega & H \\ - & \sigma \\ \hline u & \alpha \\ v & \beta \\ T & \gamma \end{array} \xrightarrow[\beta_b]{tm^{uv}_w} \begin{array}{c|c} \omega & H \\ - & \sigma \\ \hline w & (\alpha+\beta) /\!\!/ \left(\begin{smallmatrix} u,v \\ \to w \end{smallmatrix}\right) \\ T & \gamma \end{array} \qquad R^{\pm}_{ux} \underset{\beta_b}{=} \begin{array}{c|c} 1 & x \\ - & t_u^{\pm 1} \\ \hline u & t_u^{\pm 1}-1 \end{array}$$

$$\begin{array}{c|ccc} \omega & x & y & H \\ - & \sigma_x & \sigma_y & \sigma \\ \hline T & \alpha & \beta & \gamma \end{array} \xrightarrow[\beta_b]{hm^{xy}_z} \begin{array}{c|cc} \omega & z & H \\ - & \sigma_x\sigma_y & \sigma \\ \hline T & \alpha+\sigma_x\beta & \gamma \end{array} \qquad \begin{array}{c|cc} \omega & x & H \\ - & \sigma_x & \sigma \\ \hline u & \alpha & \beta \\ T & \gamma & \delta \end{array} \xrightarrow[\beta_b]{sw^{ux}_{th}} \begin{array}{c|cc} \omega+\alpha & x & H \\ - & \sigma_x & \sigma \\ \hline u & \sigma_x\alpha & \sigma_x\beta \\ T & \gamma & \delta+\frac{\alpha\delta-\gamma\beta}{\omega} \end{array} =: \begin{array}{c|c} \cdot & - \\ \cdot & - \\ \hline | & \begin{pmatrix} \sigma_x & \\ & 1 \end{pmatrix} \cdot A^{ux} \end{array}$$

Note. $A^{ux} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta+\frac{\alpha\delta-\gamma\beta}{\omega} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \frac{(\omega+\alpha)\delta-\gamma\beta}{\omega} \end{pmatrix} = \frac{1}{\omega}\left[(\omega+\alpha)\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} - \begin{pmatrix} \alpha \\ \gamma \end{pmatrix}\begin{pmatrix} \alpha & \beta \end{pmatrix}\right] = \frac{1}{\omega}\left[(\omega+a_{ux})A - a_{*x}a_{u*}\right]$.

Claim. $\omega^{k-1} \mid \Lambda^k A$ and $\omega^k \mid \Lambda^{k+1}A$ implies $(\omega+\alpha)^{k-1} \mid \Lambda^k A^{ux}$. (with $\alpha = a_{ux}$)

Proof. With $\bar u \in T^k$ and $\bar x \in H^k$, ω^k divides $\begin{vmatrix} \omega & 0 \\ 0 & a_{\bar u\bar x} \end{vmatrix}$ and $\begin{vmatrix} a_{ux} & a_{u\bar x} \\ a_{\bar u x} & a_{\bar u\bar x} \end{vmatrix}$ and hence their sum, $\begin{vmatrix} \omega+\alpha & a_{u\bar x} \\ a_{\bar u x} & a_{\bar u\bar x} \end{vmatrix} = (\omega+\alpha)\begin{vmatrix} 1 & 0 \\ 0 & a_{\bar u\bar x} - \frac{1}{\omega+\alpha}a_{\bar u x}a_{u\bar x} \end{vmatrix} = \frac{1}{(\omega+\alpha)^{k-1}}\left|(\omega+\alpha)a_{\bar u\bar x} - a_{\bar u x}a_{u\bar x}\right|$. So $\frac{1}{(\omega+\alpha)^{k-1}}\left|\frac{1}{\omega}\left[(\omega+\alpha)a_{\bar u\bar x} - a_{\bar u x}a_{u\bar x}\right]\right|$ is integral. □

That is, with $A_{\bar u;\bar x}$ denoting minors, if $\omega^{k-1}\mu_{\bar u;\bar x} = A_{\bar u;\bar x}$ and $\omega^k\mu_{u\bar u;x\bar x} = A_{u\bar u;x\bar x}$, then $(\omega+\alpha)^{k-1}(\cancel{\omega}\mu_{\bar u;\bar x} + \mu_{u\bar u;x\bar x}) = A^{ux}_{\bar u;\bar x}$.

$$w^{k-1}(w\mu_{\bar u,\bar x}) = \begin{vmatrix} w & \\ & a_{\bar u\bar x} \end{vmatrix} \qquad w^k\mu_{u\bar u,x\bar x} = \begin{vmatrix} a_{ux} & a_{u\bar x} \\ a_{\bar u x} & a_{\bar u\bar x} \end{vmatrix}$$

So $w^k(\mu_{\bar u,\bar x} + \mu_{u\bar u,x\bar x}) = \frac{1}{(w+\alpha)^{k-1}}\left|(w+\alpha)a_{\bar u\bar x} - a_{\bar u x}a_{u\bar x}\right|$

So $(w+\alpha)^{k-1}(\mu_{\bar u\bar x} + \mu_{u\bar u,x\bar x}) = \left|\frac{1}{w}\left[(w+\alpha)a_{\bar u\bar x} - a_{\bar u x}a_{u\bar x}\right]\right|$

$$= (A^{ux})_{\bar u,\bar x}$$

To do. • Consider a verification program. • Add dm formulas. • Add Burau calculus. • Add the conjugation relation. • Add the MVA formula