beta formulas

March-12-13

2:26 PM

7.2. **Formal.** For a finite set T let $R = R(T) := \mathbb{Q}[\![\{c_u\}_{u\in T}]\!]$ denote the ring of power series with commuting generators c_u corresponding to the elements of T, and let $L = L(T) := R\otimes\mathbb{Q}T$ be the the free R-module with generators T. Turn L into a Lie algebra over R by declaring that $[u,v] = c_u v - c_v u$ for any $u, v\in T$. Let $c\colon L\to R$ be the R-linear extension of $u\mapsto c_u$; namely, $\gamma = \sum_u \gamma_u u\in L\mapsto c_\gamma := \sum_u \gamma_u c_u$, where the γ_u's are coefficients in R. Note that with this definition we have $[\alpha,\beta] = c_\alpha\beta - c_\beta\alpha$ for any $\alpha,\beta\in L$. There are obvious ~~surjections~~ maps $\pi\colon FL\to L$ and $\pi\colon CW\to R$.

Lemma-Definition 7.2. The operations bch, C_u, RC_u, div_u, and J_u descend from FL/CW rearrange to L/R, and, for $\alpha,\beta,\gamma\in L$ (with $\gamma = \sum_v \gamma_v v$) they are given by

$$
\begin{aligned}
\mathrm{bch}(\alpha,\beta) &= \frac{c_\alpha + c_\beta}{e^{c_\alpha + c_\beta} - 1}\left(\frac{e^{c_\alpha}-1}{c_\alpha}\alpha + e^{c_\alpha}\frac{e^{c_\beta}-1}{c_\beta}\beta\right), && \text{betabch}\\
\checkmark\ -\gamma\quad v /\!\!/ C_u^\gamma &= v /\!\!/ RC_u^\gamma = v \quad \text{for } u\neq v\in T, && \text{betaonV}\\
\checkmark\ -\gamma\quad \rho /\!\!/ C_u^\gamma &= \rho /\!\!/ RC_u^\gamma = \rho \quad \text{for } \rho\in R, && \text{beta on rho}\\
\checkmark\ -\gamma\quad u /\!\!/ C_u^\gamma &= \mathit{MORE}, \quad \text{below} && \text{betaC}\\
u /\!\!/ RC_u^\gamma &= \left(1 + c_u\gamma_u\frac{e^{c_\gamma}-1}{c_\gamma}\right)^{-1}\left(e^{c_\gamma}u - c_u\frac{e^{c_\gamma}-1}{c_\gamma}\sum_{v\neq u}\gamma_v v\right), && \text{betaRC}\\
\mathrm{div}_u\,\gamma &= c_u\gamma_u, && \text{betadiv}\\
J_u(\gamma) &= \log\left(1 + \frac{e^{c_\gamma}-1}{c_\gamma}c_u\gamma_u\right). && \text{betaJ}
\end{aligned}
$$

$$\mathrm{ad}(-\gamma)(u) = -[\gamma,u] = -c_\gamma u + c_u\gamma =: \delta$$

$$\mathrm{ad}(-\gamma)(\gamma) = 0 \qquad e^{-\mathrm{ad}\gamma}\gamma = \gamma$$

$$\mathrm{ad}(-\gamma)(\delta) = -c_\gamma\delta \qquad e^{-\mathrm{ad}\gamma}\delta = e^{-c_\gamma}\delta$$

$$e^{-\mathrm{ad}\gamma}(u) = e^{-\mathrm{ad}\gamma}\left(-\frac{\delta}{c_\gamma} + \frac{c_u}{c_\gamma}\gamma\right)$$

$$= -\frac{e^{-c_\gamma}}{c_\gamma}\delta + \frac{c_u}{c_\gamma}\gamma$$

$$= e^{-c_\gamma}u + \frac{c_u}{c_\gamma}\left(1 - e^{-c_\gamma}\right)\gamma$$

So more $= e^{-c_\gamma}u - c_u\dfrac{e^{-c_\gamma}-1}{c_\gamma}\gamma$

$$= e^{-c_\gamma}\left(u + c_u\frac{e^{c_\gamma}-1}{c_\gamma}\gamma\right) \qquad \gamma = \sum\gamma_v v$$

$$= e^{-c_\gamma}\left[\left(1 + c_u\gamma_u\frac{e^{c_\gamma}-1}{c_\gamma}\right)u + \sum_{v\neq u} c_u\frac{e^{c_\gamma}-1}{c_\gamma}\gamma_v v\right]$$

$$u \xrightarrow{RC_u^\gamma} (\qquad)^{-1}\left(e^{c_\gamma}u - c_u\frac{e^{c_\gamma}-1}{c_\gamma}\sum_{v\neq u}\gamma_v v\right)$$

$$\xmapsto{C_u^{-\gamma}} u + (\qquad)^{-1}\left[e^{c_\gamma}e^{-c_\gamma}\sum_{v\neq u} c_u\frac{e^{c_\gamma}-1}{c_\gamma}\gamma_v v - \dots\right] = u$$

$$\xmapsto{C_u^{-\gamma}} u + (\quad)^{-1}\left[\ell^{c_\gamma}\ell^{-c_\gamma}\sum_{v \neq u} c_u \frac{\ell^{c_\gamma - 1}}{c_\gamma} \gamma_v v - \ldots\right] = u$$