It is the universal solution to a topological problem and it has many siblings (who talk to each other).
It is explicitly computable. Its target space is in itself a space of "universal formulas in Lie algebras" (that’s "the miracle"). It seems to be a complete(?) evaluation a certain gauge theory. It is related to a deep conjecture in Lie theory proven by Alekseev and Meinrenken. It has even-better-computable
specializations, including one which is an "ultimate Alexander invariant". And plenty of work remains to be done.

Trees and Wheels and Balloons and Hoops: Why I Care

The Invariant $\zeta$, Set $(\zeta_0) = (x \to 0; 0)$, \( \zeta(x) = (\zeta_0; 0) \), and $\zeta \bigl( \frac{1}{n^2}; 0 \bigr) 

Theorem. $\zeta$ is (the log of) the unique homomorphic universal finite type invariant of $\text{Spin}^c$ manifolds.

$
\zeta$ is computable! $\zeta$ of the Borromean tangle, to degree 3.

Tensorial Interpretation. Let $g$ be a finite dimensional Lie algebra (any!). Then there's a $\tau : FL(T) \to \text{Fun}(\mathcal{O}(g) \to g)$ and $\tau : CW(T) \to \text{Fun}(\mathcal{O}(g) \to g)$, Together, $\tau : M(T; H) \to \text{Fun}(\mathcal{O}(g) \to g)$, and hence $e^\tau : M(T; H) \to \text{Fun}(\mathcal{O}(g) \to \text{Lie}(\mathcal{O}(g)))$.

$\zeta$ and BF Theory. (See Cattaneo-Rossi, arXiv:math-ph/0210037) Let $A$ denote a $g$-connection on $S^3$ with curvature $F_A$, and $B$ a $g$-valued 2-form on $S^3$. For a hook $\gamma_A$, let $\text{hol}_{\gamma_A}(A) \in \mathcal{O}(g)$ be the holonomy of $A$ along $\gamma_A$. For a ball $\gamma_B$, let $\text{hol}_{\gamma_B}(B) \in g^*$ be the integral of $B$ (transported via $A$ to $\gamma_B$).

Loose Conjecture. For $\gamma \in K(T; H)$:

$$
\int DADT e^{iB(a, b)} \prod e^{\alpha_B(a, b)} \text{hol}_{\gamma_B}(A) = e^{i(\zeta(A))}.
$$

That is, $\zeta$ is a complete evaluation of the BF TQFT.

Questions. How exactly is $B$ transported via $a$? How does the ribbon condition arise? Or if it doesn't, could it be that $K$ can be generalized??

The $\beta$ quotient. $\lambda$ Arises when $g$ is the 2D non-Abelian Lie algebra.

Arises when reducing by relations satisfied by the weight system of the Alexander polynomial.

"God created the knots, all else in topology is the work of mortals."

Paper in progress: ca35/kbh Class next year: ca5/1550/1620

The $\beta$ quotient.

Let $H = \mathbb{Q}[[c]]/\mathbb{Z}$ and $L_\beta := R \otimes T$ with central $R$ and with $[u, v] = c_0 v - c_1 u$ for $u, v \in R$. Then $F_L \to L_\beta$ and $CW \to R$. Under this, $W \to X$ with $\lambda = \sum \lambda_i \chi_i$, $\lambda_i, \omega \in R$.

$\zeta(W) = \sum \lambda_i \chi_i$.

Therefore $\lambda \to \lambda + \chi_i$.

If $\lambda = \sum \lambda_i \chi_i$ then with $c_i := \sum c_i \cdot \chi_i$.

so $\zeta$ is formula-computable to all orders! Can we simplify?

Repackaging.

Given ($(\alpha_{\gamma_{\lambda}}); \omega$), set $c_i := \sum c_i \alpha_{\gamma_{\lambda}}$, replace $\alpha_{\gamma_{\lambda}} \rightarrow \alpha_{\gamma_{\lambda}} - c_i \cdot \omega$ and $\omega \rightarrow \omega - c_i \cdot \omega$, use $t_n = e^{c_i}$, and write $\alpha_{\gamma_{\lambda}}$ as a matrix. Get $\beta$ calculus.

$\beta$ Calculus. Let $\beta(H, T)$ be

$$
\begin{cases}
\alpha x & \beta y \\
\alpha z & \beta 0
\end{cases}
$$

and the $\alpha_{\gamma_{\lambda}}$'s are rational functions in variables $t_n$ one for each $n \in T$.

On long knots, $\omega$ is the Alexander polynomial!!

Why we care! $\mathcal{A}$. An ultimate Alexander invariant: Manifestly polynomial (time and space) extension of the (multivariable) Alexander polynomial to tangles. Every step of the computation is the computation of the invariant of some topological thing (no fishy Gaussian elimination). If there should be an Alexander invariant which is an algebraic categorification, it is this one.

See also ca35/regina, ca5/gwen.

Repackaging.

Restricted to $A, K, \text{ rank } K$, and $K$ should have vanishing Alexander and the Alexander polynomial.

See also ca35/kho, ca5/gwen, ca5/swi.

Tennessee, SwissKnots/Strasbourg, Bonn