Jones on "Etingof - Kazhdan quantization of Lie bialgebras I"

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5:04 PM

http://www.newton.ac.uk/programmes/GDO/seminars/2013021910301.html

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Video of talk in Newton Inst, Feb 19 2013

John Jones on E-K quantization of Lie bialgebras, following E-K I, II, 96, 98
also Arxiv / E-K
also Geer
also Kassel

also Drinfeld's ICM notes.

Lie Bialgebras Field $\mathbb{k}$ of char $= 0$

A Lie ~~alg~~ bialgebra L over $\mathbb{k}$:

1. $[\ ,\ ]$
2. $\gamma : L \longrightarrow L \otimes L$ "cobracket"

s.t.

(a) γ is a 1-cocycle on L with coefs in $L \otimes L$:

L acts on $L \otimes L$ as tensor prod of adjoint reps.

$$\gamma([x,y]) = (ad_x \otimes 1 + 1 \otimes ad_x)\gamma(y) - (ad_y \otimes 1 + 1 \otimes ad_y)\gamma(x)$$

(b) the dual of γ, $\gamma^* : L^* \otimes L^* \to L^*$ is a Lie bracket.

Example: Let G be a Poisson Lie group so $\{\ ,\ \}$ on $C^\infty(G)$, w/

$$\{f,g\}(xy) = \{f \circ L_x, g \circ L_x\}(y) + \{f \circ R_y, g \circ R_y\}(x)$$

Poisson tensor: $P \in \Lambda^2 TG$

$\{f,g\} = \langle df \wedge dg, P \rangle$

In terms of P, the eqns are

$P(x,y) = L_x(P(y)) + R_y(P(x))$

Note that $\{f,g\}(e) = 0$ → Never comes from a symplectic structure!

$\{,\}$ defines a Lie bracket on L^*,

Where L is the Lie alg of G:

$f,g \in C^\infty(G)$ $\quad \xi = (df)_e \in L^*$

$\eta = (dg)_e \in L^*$

$[\xi, \eta] = (d\{f,g\})_e$

Using right translations to trivialize TG

$P \to p: G \to \Lambda^2(L) \subset L \otimes L$

$\gamma = (dp)_e : L \to \Lambda^2(L)$ $\qquad$ 18:56

Quantized universal enveloping algebras (QUEA)

Topologically free modules of $K = \mathbb{K}[[h]]$

M K-module $\quad M/h^m M \to M/h^{m+1}M \dots$

$\hat{M} = \varprojlim M/\hbar^n M$

M is a topological K-module if $M = \hat{M}$.

$M \otimes N := \widehat{M \otimes_K N}$

Topologically free means $M \cong V[[\hbar]]$

$V[[\hbar]] \otimes W[[\hbar]] = (V \otimes W)[[\hbar]]$

H is a topological Hopf algebra over K:

1. product $\mu: H \otimes H \to H$
2. coproduct: $\Delta: H \to H \otimes H$
3. unit $\eta: K \to H$
4. co-unit $\varepsilon: H \to K$
5. antipode $S: H \to H$

... subject to the usual conditions.

H is a top Hopf algebra $\to$

$B = H/\hbar H$ is a Hopf algebra over $\mathbb{K}$

H is a "formal deformation" of B.

a QUEA is a formal def. of ~~$U(g)$~~ $U(L)$

This makes L into a Lie bialg (Drinfel'd)

$x \in L \rightsquigarrow \tilde{x} \in H$ s.t. $\tilde{x} = x \mod \hbar$

$\delta(x) := \left(\frac{1}{\hbar}(\Delta(\tilde{x}) - \Delta^{op}(\tilde{x}))\right) \mod \hbar$

Manin Triples

A ~~Mon~~ Manin triple is (L, L^+, L^-) s.t.

1. L is a Lie alg.
3. L^+, L^- are subalgebras, isotropic
2. a non-degenerate inner product $\langle\,,\,\rangle$.

s.t. $L \cong L^+ \oplus L^-$ So L^+ is dual to L^- & L^- is dual to L^+

So L^+ is a Lie-bialg. (& also L^-)

Proposition Let (L, μ, δ) be ~~a~~ a Lie bialg & (L^*, δ^*, μ^*) its dual, then there is a unique Lie alg structure on $L \oplus L^* = D$ s.t. $(\underbrace{L \oplus L^*}_{= D}, L, L^*)$ is a Manin triple.

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Proof . . . . .

$$[x,y] = [x,y]_L$$
$$[\xi,\eta] = [\xi,\eta]_{L^*}$$
$$[x,\xi] = -ad^*_\xi(x) + ad^*_x(\xi)$$

D has a Lie bialgebra structure:

$$L\oplus L^* \longrightarrow (L\otimes L)\oplus(L^*\otimes L^*)$$
$$\delta_D = \delta_L \oplus (-\mu_L)^*$$

"a quasi-triangular Lie bialg" —
the cocycle δ_D is the coboundary of r, which satisfies CYBE. 46:24

Continued on March 15, 2013:

Associators and the Drinfeld Category

Let T_n be the algebra over $\mathbb{k}$:

$$T_n = \langle t_{ij} \rangle \Big/ \begin{array}{l} t_{ij} = t_{ji} \\ [t_{ij}, t_{lm}] = 0 \end{array}$$

$[t_{ij}, t_{ik}+t_{jk}] = 0$

"the infinitesimal braid algebra"

50:00–51:00 A comment on formality of this algebra on the homological side

Let $P_1 \dots P_n$ be disjoint subsets of $\underline{m}$,

$$\rho_{P_1\dots P_n} : T_n \longrightarrow T_m$$

by

$$\rho_{P_1\dots P_n}(t_{ij}) = \sum_{\substack{p\in P_i \\ q\in P_j}} t_{pq}$$

write for $X \in T_n$:

$$X_{P_1\dots P_n} := \rho_{P_1\dots P_n}(X)$$

why $\hbar$?

$\Phi \in T_3[[\hbar]]$ is an associator if

$$\Phi_{1,2,34}\,\Phi_{12,3,4} = \Phi_{2,3,4}\,\Phi_{1,23,4}\,\Phi_{1,2,3} \quad \text{in } T_4$$

(the pentagon)

Set $B = e^{\hbar t_{12}/2} \in T_2$

...

Hexagon:

$$B_{12,3} = \Phi_{3,1,2} B_{1,3} \Phi^{-1}_{1,3,2} B_{2,3} \Phi_{123}$$

$$B_{1,23} = \Phi^{-1}_{2,3,1} B_{1,3} \Phi_{2,1,3} B_{12} \Phi^{-1}_{123}$$

in T_3

(the hexagons)

Now take $P \in \widehat{Lie}(X,Y)$ & set

$$\Phi = e^{P(\hbar t_{12}, \hbar t_{23})} \in T_3[[\hbar]]$$

Φ is a "Lie associator" if such P exists [& pent & hex hold]

Let L be a f.d. Lie bialg w/ double D.

Let $\Omega \in S^2(D)$ be

$$\Omega = \sum (l_i \otimes l^i + l^i \otimes l_i)$$

$$\Omega_{ij} \in \mathrm{End}(V_1 \otimes V_2 \otimes V_3) \qquad i,j \in \underline{3}$$

Let $\mathcal{M}$ be the category w/ objects D-modules, yet w/

$\mathrm{Hom}_{\mathcal{M}}(U, V) = \mathrm{Hom}_{\mathcal{J}}(U, V)[[\hbar]]$

This is a $\mathbb{K}[[\hbar]]$-additive category

Use the associator to make it a braided monoidal category.

Given $V_{1,2,3}$ in $\mathfrak{D}$-mod,

$$\Omega_{ij} \in \mathrm{End}_{\mathcal{M}}(V_1 \otimes V_2 \otimes V_3)$$

Consider $\Theta : T_3[[\hbar]] \to \mathrm{End}(V_1 \otimes V_2 \otimes V_3)$ in the obvious way: $\Theta(t_{ij}) = \Omega_{ij}$

$$\Phi_{V_1 V_2 V_3} := \Theta(\Phi)$$

$$\beta_{V_1 V_2} := \underset{\substack{\uparrow \\ \text{switch}}}{S} \circ e^{\hbar \Omega/2}$$

<u>Thm</u> (Drinfeld) Φ & β make $\mathcal{M}$ a braided monoidal category.