1. Calegoro-Moser systems & Dunkl operators
2. Cherednik algebras (CA)
3. Representations of CA
4. Supports of representations
5. Coloured Homfly polynomials of knots
6. Coloured Homfly of torus knots and representations of CA
7. Connections to \( \mathbb{D} \)-modules, singularities, ab. geom.

\[ \begin{array}{c}
\bigotimes_k \\
\bigoplus \end{array} \rightarrow \rho_k(a, a, k) \]

\( T(2, 3) \) partition

\( P(a, a, k) \) has non-negative costs if \( k \)

is a torus knot. \( T(p, q) \)

\[(x^p = y^q) \cap S^3 \subset \mathbb{C}^2 \]

CM particles particles on a line, w/ interaction potential \( \frac{k}{(x_i - x_j)^2} \):

\[ H_c = \frac{1}{2} \left( \sum_{i} p_i^2 + k \sum_{i \neq j} \frac{1}{(x_i - x_j)^2} \right) \]

An integrable system! \( \text{(Moser 1975)} \)
The quantum version:

\[ H = \frac{1}{2}(\Delta_0 + \sum_{j=1}^{N} \frac{1}{(x_j - x_0)^2}) \quad \Delta = \sum_{j} \Delta_j \]

\[ L_2 = -\frac{1}{2} H = \ldots \]

Thm (Mois, Olshanetsky-Perelomov)

H defines a quantum integrable system -

\[ \prod_{i \neq j} L_i L_j \]

diff. ops.

s.t. \[ L_k = \sum \Delta_i e^k + \text{lower order terms} \]

and \[ \prod_{i \neq j} L_i L_j = 0 \quad (w/ \text{above}) \]

Is this like the existence of an expansion?

\[ L_2 \Psi = \lambda \Psi \quad \text{many quantum number} \quad \rightarrow \text{reduces to ODE's.} \]

Nicest proof: Use Dunkl operators:

\[ D_i = \partial_i + c \sum_{i \neq j} \frac{\sin \frac{x_i - x_j}{c}}{(x_i - x_j)^2} \quad k = c(\text{CH}) \]
\[ \text{Prop } [D_i, D_j] = 0 \]

Now \[ L_k = \sum_{i=1}^{n} D_i \text{}^k \text{ symmetric functions } \rightarrow \text{ these are diff. ops.} \]
\[ \text{will } L_2 \text{ as before} \]

What algebraic structure is formed by \( D_i \), \( x_i \), \( S_n \)?

**Def.** The algebra generated by these is called the rational Cherednik algebra \( H_c(n) \).

\[ (D_i) \]

\[ \text{Ans. } \{ x_i : x_i \in S_n \}, y_i \]

\[ \forall i,j: [x_i, x_j] = [y_i, y_j] = 0 \]

\[ s_i x_i s_i^{-1} = y_{si}, \quad s_i y_i s_i^{-1} = y_{si} \]

\[ i \neq j: [y_i, x_j] = c_{ij}, \quad [y_i, x_i] = \frac{1 - \sum c_{ij} s_{ij}}{\text{deg}} \]

Has "Onakel op. rep" on \( C[x_1, \ldots, x_n] \)

**Has PBW Thm:**

\[ C[x_1, \ldots, x_n] \otimes C_{S_n} \otimes C[y_1, \ldots, y_n] \rightarrow H_c(n) \]

is an isomorphism.
\[ \nabla_+ = \frac{\partial}{\partial y} \]

Can define:

1. Category \( \mathcal{C}_n \): rep. of \( H_c(n) \) which are finitely generated over \( \mathbb{C}[x] \) and locally nilpotent under \( y_1, \ldots, y_n \)

2. Verma modules: \( \chi \)-partition \( (n) \)

\[ \Pi_{\chi} : \text{corresp. rep. of } S_n \]

\[ M_c(\chi) := H_c(n) \otimes_{S_n} \mathcal{C}(y_1, \ldots, y_n) / \Pi_{\chi} \]

\[ = \mathbb{C}[x_1, \ldots, x_n] \otimes \Pi_{\chi} \]

\( y_i \) acts as \( D_i^\chi := 2_i + \sum_{j \neq i} \frac{(s_{ij} - 1)}{x_i - x_j} S_{ij} \)

Irreps: \( L_c(\chi) = \text{irred. quo of } M_c(\chi) \)

These are \( \mathbb{Z} \)-graded.

Problem: Compute characters of \( L_c(\chi) \)

(for SS alg. \rightarrow Knizhnik-Zamolodchikov polys)

CA \( \Rightarrow \) special case of affine KZ polynomials.
\[ \text{Supp}(M) = Z(\text{Ann}(M)) \subset C^n \]

Then (Berz., Etin.) The only possible supports for \( M \in \mathcal{O}_C(n) \) are:

--- See video ---

Can divide partitions by numbers 1

--- More in video ---

\[ \text{Thm} \quad P_x(q, -\alpha; T(m, \frac{m}{2}, 0)) = \]

HomFLY

\[ \sum_{k=0}^{n} \text{ch} \ \text{Hom}_{\mathcal{O}_C}(\Lambda^k C^n, L_{m, (\frac{m}{2}, 0)}(\eta \chi)) \alpha^k \]

Cor 1. LHS has pos. degree

Cor 2. RHS is symmetric under mean.