Etingof @ Colloq: Cherednik algebras and torus knots

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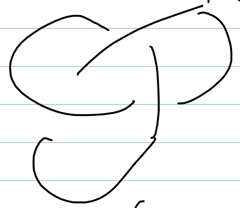
1. Calogero-Moser (CM) systems & Dunkl operators
2. Cherednik algebras (CA)
3. Representations of CA
4. Supports of representations
5. Coloured HOMFLY polynomials of knots.
6. Coloured HOMFLY of torus knots and representations of CA.
7. Connections to $\mathcal{D}$-modules, singularities, alg. geom.

K



$\longrightarrow P_\lambda(q, a, K)$

partition

$T(2,3)$

$P_\lambda(q, -a, K)$ has non-negative coeffs if K is a torus knot. $T(p,q)$

$$\left((x^p = y^q) \cap S^3 \right) \subset \mathbb{C}^2$$

CM particles particles on a line, w/ interaction potential $\frac{k}{(x_1 - x_2)^2}$:

$$H_{cl} = \frac{1}{2}\left(\sum_{i=1}^{n} p_i^2 + k \sum_{i \neq j} \frac{1}{(x_i - x_j)^2} \right)$$

An integrable system! (Moser 1975)

The quantum version:

$$H = \frac{1}{2}\left(-\Delta + K\sum_{i\neq j}\frac{1}{(x_i-x_j)^2}\right) \qquad \Delta = \sum \partial_i^2$$

$$L_2 = -\frac{1}{2}H = \dots$$

Thm (Moser, Olshanetsky–Perelomov)

H defines a quantum integrable system –

$\exists\ \underbrace{L_1 \quad L_n}_{\text{diff. ops.}}$ of hom. degree $-1, -2, -3, \dots$

s.t. $L_k = \sum \partial_i^k$ + lower order terms

and $[L_i, L_j] = 0$ (w/ L_2 as above)

Is this like the existence of an expansion?

$L_2\Psi = \lambda\Psi \longrightarrow$ many $\overset{\text{conserved}}{\text{quantum number}}$

$\longrightarrow$ reduces to ODE's.

Nicest proof: Use Dunkl operators:

$$D_i = \partial_i + c\sum_{i\neq j}\frac{s_{ij}}{x_i - x_j} \qquad K = C(C+1)$$

Prop $[D_i, D_j] = 0$

Now $L_k = \sum_{i=1}^{n} D_i^k \Big|_{\text{symmetric functions}} \longrightarrow$ these are diff. ops., w/ L_2 as before

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What algebraic structure is formed by D_i, x_i, S_n?

Def The algebra generated by these is called "the rational Cherednik algebra" $H_c(n)$.

gens & rels: $\langle x_i, s \in S_n, \overset{(D_i)}{\overset{\shortparallel}{y_i}} \rangle / \text{rels}$:

rel: $[x_i, x_j] = [y_i, y_j] = 0$

$s x_i s^{-1} = y_{s(i)} \qquad s y_i s^{-1} = y_{s(i)}$

$i \neq j \quad [y_i, x_j] = c_{ij} \qquad [y_i, x_i] = 1 - \sum_{j \neq i} c \cdot s_{ij}$

Has "Dunkl oper. rep" on $\mathbb{C}(x_1 \ldots x_n)$

Has PBW thm:

$$\underbrace{\mathbb{C}[x_1 \ldots x_n]}\otimes\underbrace{\mathbb{C}S_n}\otimes\underbrace{\mathbb{C}[y_1 \ldots y_n]} \to H_c(n)$$

is an isomorphism.

$\check{n}_-$ cartan n_+

$\mathfrak{h}$

Can define:

1. Category $\mathcal{O}_c(n)$: rep. of $H_c(n)$ which are finitely generated over $\mathbb{C}[x]$ and locally nilpotent under $y_1 \dots y_n$
2. Verma modules: $\lambda \in$ partitions(n)

   π_λ: corresp. rep of S_n

$$M_c(\lambda) := H_c(n) \otimes_{\mathbb{C}S_n \ltimes \mathbb{C}(y_1 \dots y_n)} \pi_\lambda$$

$$\simeq \mathbb{C}[x_1 \dots x_n] \otimes \pi_\lambda$$

y_i acts as $D_i^\lambda := \partial_i + c \sum_{j \neq i} \frac{(s_{ij} - 1) \otimes s_{ij}}{x_i - x_j}$

Irreps: $L_c(\lambda)$ = irred. quo of $M_c(\lambda)$

these are $\mathbb{Z}$-graded.

<u>problem</u> Compute characters of $L_c(\lambda)$

(for SS alg. $\longrightarrow$ Kaz-Lus polys)

CA $\xrightarrow{\text{Rouquier}}$ special case of affine KL polynomials.

$\mathrm{Supp}(M) = Z(\mathrm{Ann}(M)) \subset \mathbb{C}^n$

Thm (Bez.., Etin.) The only possible supports for $M \in \mathcal{O}_c(n)$ are:

— See video —

Can divide partitions by numbers!

— more in video —

Thm $P_\lambda(q, -a, T(\frac{m}{d}, \frac{n}{d})) =$

HOMFLY

$$\sum_{k=0}^{n} \left[ch\, \mathrm{Hom}_{S_n}\left(\Lambda^k \mathbb{C}^n, L_{\frac{m}{n}}\left(\tfrac{n}{d}\lambda\right)\right) \right] a^k$$

Cor 1. LHS has pos. coeffs.

Cor 2. RHS is symmetric under $m \leftrightarrow n$.