Oxford talk preps
January-19-13
7:22 AM

To write up a “highlights” sequence.

Balloons and Hoops and their Universal Finite-Type Invariant,
BF Theory, and an Ultimate Alexander Invariant

Scheme. • Balloons and hoops in \( R^3 \), algebraic structure and
relations with 3D.
• An ansatz for “homomorphic” invariant: computable related to
finite-type and to BF.
• Reduction to an “ultimate Alexander invariant”.

\( K^{3h}(m,n) \).

Examples. • Mean business

- Maps \( v/\omega \) to \( K^{3h} \)
- \( \delta \) is an unknotting map to \( K^{3h} \)
- \( \alpha \) contains meridians and the “overcrossing commutator” relation, and Connectivity, that's all.

Operations • Operations & Cuts / \( \sum \) • Connected

Meta-Group-Action. \( K \):

- Properties:
  - Associativity: \( m^n \circ m^m = m^{nm} \circ m^m \)
  - Action axiom 1: \( tm^a \circ hla^b = hla^b \circ tm^a \)
  - Action axiom 2: \( hla^b \circ hla^c = hla^b \circ hla^c \)
  - SD Product: \( dm^a \circ hla^b = hla^b \circ dm^a \circ hla^b \) is associative.

Thus we seek homomorphic invariants of \( K^{3h} \).

Invariant \#0. With \( H_i \) denoting “homest” \( \tau_i \), map \( \gamma \in K^{3h}(m,n) \) to the triple \( (H_1(\gamma), (u), (x_1)) \), where the meridian of the balls \( u \) normally generate \( H_0 \), and the longitudes \( x_1 \) are some elements of \( H_1 \).

- Acts like \( \ast \), \( tm \) acts by “merging” two meridians/generators, \( hm \) acts by multiplying two longitudes, and \( hla \) acts by “conjugating a meridian by a longitude”.

Failure \#0. Can we write the \( x \)'s as free words in the \( \ast \)'s?

If \( x = w \), compute \( x \circ hla^m \)

The Meta-Group-Action \( M \).

- Let \( T \) be a set of “tail labels” (“balloon colours”), and \( H \) a set of “head labels” (“hoop colours”).
- Let \( FL = I(T) \) and \( FA = H(T) \) be the (completed graded) free Lie and free associative algebras on generators \( T \) and let \( CW = CW(T) \) be the (completed graded) vector space of cyclic words on \( T \), so there's \( T : FA \to CW \).

- Let \( M(T,H) = \{(X = (x_1, \ldots, x_n)) : \lambda_1 \in FL, \omega \in CW \} \)

- Operations. Set \( (\lambda_1, \omega_1) + (\lambda_2, \omega_2) := (\lambda_1, \omega_1 + \omega_2) \) and

\[ \mu = (\lambda; \omega) \] define

- \( tm^a \circ \mu = \mu \circ (u, v \to w) \)
- \( hm^a \circ \mu = \left( x_1, y_1, \ldots, x_n, y_n \right) \cdot bch(\lambda_1, \lambda_2) \cdot \omega \]

- \( hla \circ \mu \) the “\( \ast \)space”
Balloons and Hoops and their Universal Finite-Type Invariant, 2

The Meta-Cocycle ∂ Set \( J_\partial(\lambda) := J(1) \) where \( J(0) = 0 \), \( \lambda_\sigma = \lambda \not\in CC^\lambda \), \( \frac{dJ(s)}{ds} = (J(s) \not\in \lambda_\sigma) + \div v_\lambda \lambda_\sigma \), and where \( \div v_\lambda := \text{tr}(v_\sigma\lambda(v)) \), \( \lambda_\sigma(v) = \lambda_{\alpha(v)} \sigma(v) = \lambda_{\alpha(v)}(1, \lambda_\sigma) \), \( \iota(\lambda_\sigma(\lambda)(v)) \not\in \lambda_\sigma(\lambda)(v) \), and \( v_\lambda \) is the inclusion \( FL \to FA \).

Claim. \( CC^\lambda \not\in CC^\lambda \), \( CC^\lambda \not\in CC^\lambda \), and \( J_\partial(\lambda_\sigma(\lambda)) = J_\partial(\lambda_\sigma(\lambda)) \not\in CC^\lambda \), \( J_\partial(\lambda_\sigma(\lambda)) \not\in CC^\lambda \), and hence \( tm, \tau, \text{ and } \lambda_\sigma \) form a meta-group-action.

Why ODEs? Q. Find \( f(x,y) \) s.t. \( f(x+y) = f(x)f(y) \).

A. \( \frac{d}{dx} f(x) = \frac{d}{dx} f(x) \).

Now solve this ODE using Picard's theorem or power series.

The Invariant ζ. Set \( \zeta \not\in \{ (\pm n_\sigma); 0 \} \). This at least defines an invariant of \( u/v/w \)-tangles, and if the topologists deliver a "Reidemeister theorem", it is well defined on \( K^{(u/v/w)} \).

Theorem. ζ is (the log of) a universal finite type invariant (a homomorphic expansion of \( w \))-tangles.

Tensorial Interpretation. Let \( \otimes \) be a finite dimensional Lie algebra (any!). Then there's \( \tau : FL(T) \to \text{Fun}(\otimes_T \otimes \otimes \otimes \otimes) \) and \( \tau : C(T) \to \text{Fun}(\otimes_T \otimes \otimes \otimes \otimes) \). Together, \( \tau : M(T,H) \to \text{Fun}(\otimes_T \otimes \otimes \otimes \otimes) \), and hence \( e^\tau : M(T,H) \to \text{Fun}(\otimes_T \otimes \otimes \otimes \otimes) \).

ζ and BF Theory. Let \( A \) denote a \( g \)-connection on \( S^1 \) with curvature \( F_\sigma \), and \( B \) a \( g \)-valued 2-form on \( S^3 \). For a hoop \( \gamma \), let \( h_{\gamma}(A) \in H^2 \) be the holonomy of \( A \) along \( \gamma \). For a ball \( \gamma \), let \( \Omega_\gamma(B) \in g^* \) be the integral of \( (B) \) transported via \( A \) on \( \gamma_\gamma \).

Loose Conjecture. For \( \gamma \in K(T,H) \),

\[ \int DAD\partial B Dv \prod c^{\partial v}(B) \otimes h_{\gamma}(A) = e^\tau(\gamma) \]

That is, \( \zeta \) is a complete evaluation of the BF TQFT.

How exactly is \( B \) transported via \( A \) to \( \gamma \)? How does the ribbon condition arise? Or if it doesn't, could it be that \( \zeta \) can be generalized?

The β quotient. 1. Arises when \( g \) is the 2D non-Abelian Lie algebra.

2. Arises when reducing by relations satisfied by the weight system of the Alexander polynomial.

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The β quotient. 2. Let \( R = \mathbb{Q}[c_{u/v,w}] \) and \( L_\beta := \mathbb{R} \otimes \mathbb{R} \) with \( R \) central \( R \) and with \( [u,v] = c_{u,v} - c_{v,u} \). Then \( FL \to L_\beta \) and \( CW \to R \). Under this,

\[ \mu : \lambda \Rightarrow \sum_{u,v \in T} \lambda_{u,v} \alpha_{u,v} \alpha_{v,u} \in R, \]

\[ \text{bch}(u,v) \Rightarrow c_{u,v} = \sum_{r=0}^\infty \frac{c_{u,v}}{c_{u,v}} \left( e^{c_{u,v}} - 1 \right), \]

\[ \alpha \not\in CC^\lambda \Rightarrow \sum_{r=0}^\infty \frac{c_{u,v}}{c_{u,v}} \left( e^{c_{u,v}} - 1 \right), \]

\[ \text{div}_\lambda \Rightarrow \sum_{r=0}^\infty \frac{c_{u,v}}{c_{u,v}} \left( e^{c_{u,v}} - 1 \right), \]

so \( \zeta \) is formula-computable to all orders! Can we simplify?

Rethinking. Given \( \{ (x : \lambda_{u,v}(\omega)) \}, \) set \( c_{u,v} := \sum_{r=0}^\infty \frac{c_{u,v}}{c_{u,v}} \left( e^{c_{u,v}} - 1 \right), \) and \( \omega \Rightarrow \log \omega, \) use \( h_{\gamma} \) and write \( c_{u,v} \) as a matrix. Get \( \zeta \) calculus?

Calculation. Let \( \beta(H,T) \) be

\[ \begin{bmatrix} \omega & \alpha_{u,v} & \alpha_{v,u} & \alpha_{u,v} & \alpha_{v,u} & \alpha_{u,v} & \alpha_{v,u} \\ \omega & \omega & \omega & \omega & \omega & \omega & \omega \\ \omega & \omega & \omega & \omega & \omega & \omega & \omega \\ \omega & \omega & \omega & \omega & \omega & \omega & \omega \\ \omega & \omega & \omega & \omega & \omega & \omega & \omega \\ \omega & \omega & \omega & \omega & \omega & \omega & \omega \\ \omega & \omega & \omega & \omega & \omega & \omega & \omega \end{bmatrix} \]

\[ \text{bch}(u,v) \Rightarrow \sum_{r=0}^\infty \frac{c_{u,v}}{c_{u,v}} \left( e^{c_{u,v}} - 1 \right), \]

\[ \text{div}_\lambda \Rightarrow \sum_{r=0}^\infty \frac{c_{u,v}}{c_{u,v}} \left( e^{c_{u,v}} - 1 \right), \]

on long knots, \( \omega \) is the Alexander polynomial?

Why bother? (1) An ultimate Alexander invariant: Manifestly polynomial (time and size) extension of the (multi-variable) Alexander polynomial to tangles. Every step of the computation is the computation of the invariant of some topological thing (no fishy Gaussian elimination!). If there should be an Alexander invariant to have an algebraic calculation, it is this one!

See also \ref{BFtheory}.\ref{BFtheory}

Why bother? (2) Related to \( A-T, K-V, \) and \( E \)-type, should have vast generalization beyond knots and the Alexander polynomial.

See also \ref{BFtheory}.