Deformation Quantization [From the perspective of]

Symplectic geometry

\[ M : \text{Poisson Manifold } C^\infty(M) = A \]

\[ \{ , \} : A \times A \rightarrow A \text{ multilinear} \]

Dor's comment: All this should be "unrepresented"

1. \( \{ f, g \} = -\{ g, f \} \)

2. Jacobi

3. Leibnitz

\[ \{ f \cdot g \} \cdot h = \{ f, g \} \cdot h + f \cdot \{ g, h \} \]

In the symplectic situation,

\( \omega \in \Omega^2(M, \Lambda^2 T^*M) \) makes \( \text{P} \in \Gamma(M, \Lambda^2 TM) \)

\[ \{ f, g \} = \omega(df \wedge dg) \]

A \(*\)-product is a product on \( A[[t]] \):

1. Bilinear over \( R[[t]] \).

2. Associative.

3. \( a \ast b = \sum B_n(a, b) t^n \) with

\[ B_0(a, b) = a \cdot b \quad (a, b \in A) \]

A formal deformation quantization of \( M \) is a \(*\)-product on \( A[[t]] \) s.t.

\[ \{ a, b \} = B_1(a, b) - B_1(b, a) \]

Theorem (Kontsevich) Every Poisson manifolds has a (formal) deformation quantization.

Use Gerstenhaber deformation theory:

\( V \): vector space over \( k \).
\[ C^p(V, V) := \text{Hom}(V^{\otimes p}, V) \]

Composition:
\[
(f \circ g)(v_1, \ldots, v_{p+q-1}) = f(v_1, \ldots, v_{i-1}, g(v_i, \ldots, v_{i+q-1}), v_{i+q}, \ldots, v_{p+q-1})
\]

\[ f \circ g = \sum_i \pm f \circ_i g \]

Set \[ A_L(g, h) = (f \circ g)h - f_0(g \circ h) \]
then \[ A_L(g, h) = \pm A_L(h, g) \]
so \[ [f, g] = f \circ g - (-1)^{(1f-1)(1g-1)}g \circ f \]
is a "graded Lie bracket of degree -1".

**Example:** \( M \in C^2(V, V) \) then \( M \) is associative iff \( M_0 M = 0 \) iff \( [M, M] = 0 \) \((k \neq k+2)\).

Define \( d_M : C^p \rightarrow C^{p+1} \) by
\[ d_M(x) = M_0 x \pm x_0 M = [M, x] \]
then \( d_M^2 = 0 \) - This is the Hodge-Hodshold complex of \( A = (V, M) \).

**Apply to construction of a deformation quant:***

\[ 0 : A \otimes A \rightarrow A[[t]] \]
\[ 0(a, b) = a_0 b + C(a_1 b) \]
want \[ (M + C)^0 (M + C) = 0 \]
gets the Maurer-Cartan eqns:
\[ d_M C + C \circ C = d_M C + \frac{1}{2}[C, C] = 0 \]
We're looking for solutions in 
\[ C^*(A,A)[t+t] \]
Are there solutions in 
\[ H\mathcal{H}^*(C(A,A)[t+t], d\mu) \] ?

Hochschild–Kostant–Rosenberg (HKR) Theorem:
\[ H\mathcal{H}^p(A) = \mathcal{M}(\mathcal{M}, \wedge^p T\mathcal{M}) \]
\[ \{,\} \in H\mathcal{H}^*(A,A) \leftrightarrow p \in \mathcal{M}(\mathcal{M}, \wedge^2 T\mathcal{M}) \]
\[ \{,\} \leftrightarrow \text{Schouten bracket} \]

The Schouten bracket:
1. on vector fields, \( \{\xi, \eta\} = [\xi, \eta] \)
2. \( \{\alpha, \beta \wedge \gamma\} = [\alpha, \beta] \wedge \gamma + [\gamma, \beta] \wedge \alpha \)

Classical computation: \( \{p, p\} = 0 \)
\[ dp(p) = 0 \]
So we have a solution to MC in homology.

Define \( L_1, L_2, dG_L \); A quasi-isomorphism
\( F: L_1 \rightarrow L_2 \) is a map of \( dG_L \)'s which
is an isomorphism in homology.

Claim If \( L_1 \) is quasi-iso to \( L_2 \), then
there is a solution to MC in \( L_1 \)
if and only if there is a soln in \( L_2 \)
Kontsevich Formality Theorem:
\[ C^*(A,A)[t+t] \text{ is quasi-iso to } H^*(A,A)[t+t] \]
Deligne's conjecture: see video.
(proven by McClure-Smith, Voronov, Kontsevich-
-Seidelman, . . . )