The Problem. Let $G = \langle g_1, \ldots, g_n \rangle$ be a subgroup of $S_n$ with $n = O(100)$. Before you die, understand $G$:

1. Compute $|G|$.
2. Given $\sigma \in S_n$, decide if $\sigma \in G$.
3. Write a $\sigma \in G$ in terms of $g_1, \ldots, g_n$.
4. Produce random elements of $G$.

The Commutative Analog. Let $V = \text{span}(v_1, \ldots, v_n)$ be a subspace of $\mathbb{R}^n$. Before you die, understand $V$.

Solution: Gaussian Elimination. Prepare an empty table,

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & \cdots & n-1 & n \\
\end{array}
\]

Feed $v_1, \ldots, v_n$ in order. To feed a non-zero vector, find its pivotal position $i$.

1. If box $i$ is empty, put $v_i$ there.
2. If box $i$ is occupied, find a combination $v'$ of $v$ and $u_i$ that eliminates the pivot, and feed $v'$.

Non-Commutative Gaussian Elimination

Prepare a mostly-empty table,

\[
\begin{array}{cccccc}
1 & 1 \hspace{1cm} | \hspace{1cm} 1 & 2 & 3 & 4 & \cdots & n-1 & n \\
\end{array}
\]

Feed $g_1, \ldots, g_n$ in order. To feed a non-identity $\sigma$, find its pivotal position $i$ and let $j := \sigma(i)$.

1. If box $(i, j)$ is empty, put $\sigma$ there.
2. If box $(i, j)$ contains $\sigma_{i,j}$, feed $\sigma'$ so that $\sigma' := \sigma_{i,j}^{-1} \sigma$.

The Twist. When done, for every occupied $(i, j)$ and $(k, l)$, feed $\sigma_{i,j} \sigma_{k,l}$. Repeat until the table stops changing.

Claim 1. The process stops in our lifetimes, after at most $O(n^6)$ operations. Call the resulting table $T$.

Claim 2. Every $\sigma_{i,j}$ in $T$ is in $G$.

Claim 3. Anything fed in $T$ is now a monotone product in $T$: if was fed $f \in M_1 := \langle \sigma_{1,j}, \sigma_{2,j}, \ldots, \sigma_{n,j} \rangle$; \forall i, j \geq i \& \sigma_{i,j} \in T").

Homework Problem 1: Can you do cosets?

Non-Commutative Gaussian Elimination and Rubik's Cube

\[
\begin{array}{cccccc}
1 & 1 \hspace{1cm} | \hspace{1cm} 1 & 2 & 3 & 4 & \cdots & n-1 & n \\
\end{array}
\]

The Generators

\[
\begin{array}{cccccc}
1 & 1 \hspace{1cm} | \hspace{1cm} 1 & 2 & 3 & 4 & \cdots & n-1 & n \\
\end{array}
\]

Space for a vector $u_4 \in V$, of the form $u_4 = (0, 0, 0, 1, \ldots, *)$: $1 := \text{“the pivot”}$.

Feed $v_1, \ldots, v_n$ in order. To feed a non-zero vector, find its pivotal position $i$.

1. If box $i$ is empty, put $v_i$ there.
2. If box $i$ is occupied, find a combination $v' \in V$ and $u_i$ that eliminates the pivot, and feed $v'$.

Non-Commutative Gaussian Elimination

Prepare a mostly-empty table,

\[
\begin{array}{cccccc}
1 & 1 \hspace{1cm} | \hspace{1cm} 1 & 2 & 3 & 4 & \cdots & n-1 & n \\
\end{array}
\]

Feed $g_1, \ldots, g_n$ in order. To feed a non-identity $\sigma$, find its pivotal position $i$ and let $j := \sigma(i)$.

1. If box $(i, j)$ is empty, put $\sigma$ there.
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The Twist. When done, for every occupied $(i, j)$ and $(k, l)$, feed $\sigma_{i,j} \sigma_{k,l}$. Repeat until the table stops changing.

Claim 1. The process stops in our lifetimes, after at most $O(n^6)$ operations. Call the resulting table $T$.

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Homework Problem 1: Can you do cosets?
The Back Side

A homomorphism from $S_4$ to $S_3$:

A homomorphism from $A_5$, to the symmetry group of a dodecahedron, to $A_6$: