$\sqrt{2}$ is irrational

\[ \left( \frac{p}{q} \right)^2 = 2 \]

\[ (\sqrt{2} - 1)k < k \text{ has same properties.} \]

\[ k \frac{p}{q} \in \mathbb{N} \implies k = \text{"minimal } q \text{"} \]

\[ k \left( \frac{p}{q} - 1 \right) = k \frac{p - q}{k} = 1 - q \in \mathbb{N} \]

\[ \frac{p}{q} = \frac{2q - p}{p - q} \quad \frac{p}{q} (p - q) = \frac{p^2 - q^2}{q} \]

\[ = p (\frac{p}{q} - 1) = q (\frac{p^2}{q^2} - \frac{p}{q}) \]

If \( \left( \frac{p}{q} \right)^2 = 2 \), then also \( q (2 - \frac{p}{q}) = 2q - p \)

\[ \left( \frac{2q - p}{p - q} \right)^2 = 2 \quad \text{but} \quad \left( \frac{p}{q} \right) < 2 = 7 \text{ p < 2q} \]

\[ \implies p - q < q \]

So \( \frac{p}{q} \) isn't smallest.

\[ \frac{2 - \frac{p}{q}}{\frac{p}{q} - 1} = \frac{4 - 4\frac{p}{q} + 2}{2 - 2\frac{p}{q} + 1} \]

\[ = \frac{6 - 4\frac{p}{q}}{3 - 3\frac{p}{q}} = 2 \]

If \( x - \frac{p}{q} = \sqrt{2} \), then same for \( \frac{x + 2\sqrt{x}}{2} \)

\[ \frac{x + 2\sqrt{x}}{2} = \frac{x}{2} + \frac{1}{2} = \frac{p}{2q} + \frac{q}{p} = \frac{p^2 + 2q^2}{2pq} \]

Added Jan 13, Proof by Rich Schurte:

\[ (\sqrt{2} + 1)(\sqrt{2} - 1) = 1 \]
\[(\sqrt{2} + 1)(\sqrt{2} - 1) = 1\]

\[\frac{\sqrt{2}}{2} \quad \frac{\sqrt{2}}{p} \quad \text{yet } k\sqrt{2} + 1 \text{ have the same denominator as they differ by an integer.}\]