Reduction mod/to β

$[u,v]=C_uv-C_vu$

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10:58 AM

$M(T,H):=\{\mu=(\omega,\lambda=(\lambda_x)_{x\in H}):\ \omega\in CW,\ \lambda_x\in FL\}$ 1

(e.g., $\mu=\ldots$).
With $\mu=(\omega,\lambda)$ define

$$tm^{uv}_w:\mu\mapsto\mu\,/\!/\,(u,v\mapsto w),\quad 2$$

$$hm^{xy}_z:\mu\mapsto\left(\omega,\left(\ldots,\widehat{\lambda_x},\widehat{\lambda_y},\ldots,\mathrm{bch}(\lambda_x,\lambda_y)_z\right)\right)\quad 3$$

$$hta^{xu}\ 4\ \mu\mapsto\underbrace{\overbrace{\mu\,/\!/\!/\!/}^{\text{“stable apply”}}(u\mapsto e^{\mathrm{ad}\,\lambda_x}(\bar u))\,/\!/\,(\bar u\mapsto u)}_{\mu/\!/CC_u^{\lambda_x}}+\underbrace{(J_u(\lambda_x),0)}_{\text{the “}J\text{-spice”}}$$

The Meta-Cocycle J. Set $J_u(\lambda):=J(1)$ where

div 6 $\quad J(0)=0,\quad \boxed{\lambda_s=\lambda\,/\!/\,CC_u^{s\lambda},}$ 5

$$\frac{dJ(s)}{ds}=(J(s)\,/\!/\,\mathrm{der}(u\mapsto[\lambda_s,u]))+\mathrm{div}_u\,\lambda_s.\quad 7$$

Claim.

$$CC_u^{\mathrm{bch}(\lambda_1,\lambda_2)}=CC_u^{\lambda_1}\,/\!/\,CC_u^{\lambda_2/\!/CC_u^{\lambda_1}},$$

$$J_u(\mathrm{bch}(\lambda_1,\lambda_2))=J_u(\lambda_1)\,/\!/\,CC_u^{\lambda_2/\!/CC_u^{\lambda_1}}+J_u(\lambda_2\,/\!/\,CC_u^{\lambda_1}),$$

and hence tm, hm, and hta form a meta-group-action.

1 λ_x becomes $\sum_{u\in T}\lambda_{ux}u$ so $\bar\lambda=\sum_{x,u}\lambda_{ux}u\cdot x$,

where $\lambda_{ux}\in\mathbb{Q}[\![\{C_u\}_{u\in T}]\!]=:R$

2 tm is unchanged and matches β-calculus.

3 $$\mathrm{bch}(u,v)=\frac{e^{C_u}-1}{C_u}\cdot\frac{C_u+C_v}{e^{C_u+C_v}-1}u+e^{C_u}\frac{e^{C_v}-1}{C_v}\frac{C_u+C_v}{e^{C_u+C_v}-1}v$$

$$\frac{(-1+e^{\alpha c_1})\ (\alpha c_1+\beta c_2)}{(-1+e^{\alpha c_1+\beta c_2})\ c_1}$$

$$\frac{e^{\alpha c_1}\ (-1+e^{\beta c_2})\ (\alpha c_1+\beta c_2)}{(-1+e^{\alpha c_1+\beta c_2})\ c_2}$$

4 If $\lambda=\sum_v\lambda_v v$; set $C_\lambda=\sum\lambda_v C_v\in R$. Then

$[\lambda,\bar u]=C_\lambda\bar u-C_{\bar u}\lambda$ i.e. $\mathrm{ad}\lambda$: $\bar u\xrightarrow{C_\lambda}\bar u$, $\bar u\xrightarrow{-C_{\bar u}}\lambda\to 0$

so $$e^{\mathrm{ad}\lambda}(\bar u)=e^{C_\lambda}\bar u-C_{\bar u}\frac{e^{C_\lambda}-1}{C_\lambda}\overbrace{\left[\lambda_u u+\sum_{v\neq u}\lambda_v v\right]}^{\lambda}$$

so if $\lambda=\lambda_u u+\sum_{v\neq u}\lambda_v v$ then

$$u/\!/\!/\!/(u\to e^{\mathrm{ad}\lambda}(\bar u))=$$

$$\sum_{k=0}^{\infty}\left(-C_{\bar u}\lambda_u\frac{e^{C_\lambda}-1}{C_\lambda}\right)^k\left(e^{C_\lambda}\bar u-C_{\bar u}\frac{e^{C_\lambda}-1}{C_\lambda}\sum_{v\neq u}\lambda_v v\right)$$

$$\sum_{k=0}\left(-c_{\bar u}\lambda_u\frac{e^{c_\lambda}-1}{c_\lambda}\right)\left(e^{-c_\lambda}\bar u-c_{\bar u}\frac{e^{c_\lambda}-1}{c_\lambda}\sum_{v\neq u}\lambda_v v\right)$$

So $u // CC_u^{\lambda} = u //// (u \to e^{\alpha\lambda}(\bar u)) // \bar u \to u =$

$$\left(1+c_u\lambda_u\frac{e^{c_\lambda}-1}{c_\lambda}\right)^{-1}\left(e^{c_\lambda}u-c_u\frac{e^{c_\lambda}-1}{c_\lambda}\sum_{v\neq u}\lambda_v v\right)$$

$$\begin{pmatrix}
\omega\left(1+\frac{(-1+e^{\alpha c_1+\gamma c_2})\,\alpha c_1}{\alpha c_1+\gamma c_2}\right) & h[1] & h[2] \\
t[1] & \frac{e^{\alpha c_1+\gamma c_2}\,\alpha\,(\alpha c_1+\gamma c_2)}{e^{\alpha c_1+\gamma c_2}\,\alpha c_1+\gamma c_2} & \frac{e^{\alpha c_1+\gamma c_2}\,\beta\,(\alpha c_1+\gamma c_2)}{e^{\alpha c_1+\gamma c_2}\,\alpha c_1+\gamma c_2} \\
t[2] & \frac{\gamma\,(\alpha c_1+\gamma c_2)}{e^{\alpha c_1+\gamma c_2}\,\alpha c_1+\gamma c_2} & \frac{-\beta\gamma+\alpha\delta+\frac{\beta\gamma\,(\alpha c_1+\gamma c_2)}{e^{\alpha c_1+\gamma c_2}\,\alpha c_1+\gamma c_2}}{\alpha}
\end{pmatrix}$$

$\alpha u+\gamma v // C_u^{\alpha u+\gamma v} =$

$$=\left(1+c_u\alpha\frac{e^{c_u\alpha+c_v\gamma}-1}{c_u\alpha+c_v\gamma}\right)^{-1}\left(e^{c_u\alpha+c_v\gamma}u-\overbrace{\;\;}^{c_u\frac{e^{c_u\alpha+c_v\gamma}-1}{c_u\alpha+c_v\gamma}}\gamma v\right)\cdot\alpha+\gamma v$$

$$=\left(\frac{\cancel{c_u\alpha}+c_v\gamma+c_u\alpha\cdot e^{c_u\alpha+c_v\gamma}-\cancel{c_u\alpha}}{c_u\alpha+c_v\gamma}\right)^{-1}\left(\quad\right)\alpha+\gamma v$$

$$=\left(\frac{c_u\alpha+c_v\gamma}{c_v\gamma+c_u\alpha e^{c_u\alpha+c_v\gamma}}\right)\left(e^{c_u\alpha+c_v\gamma}u-\underbrace{\;\;}_{c_u\frac{e^{c_u\alpha+c_v\gamma}-1}{c_u\alpha+c_v\gamma}}\gamma v\right)\alpha+\gamma v$$

The coefficient of u matches. The coff of v is:

$$\frac{-\gamma c_u\left(e^{c_u\alpha+c_v\gamma}-1\right)\cdot\alpha}{c_v\gamma+c_u\alpha e^{c_u\alpha+c_v\gamma}}+\gamma=\gamma\frac{\alpha c_u+\gamma c_v}{\text{den}}$$

matches too!

Need to also check $\beta u + \delta v \,//\, C_u^{\alpha u + \gamma v}$ but I'll skip.

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5 If $\lambda = \lambda_u u + \sum_{v \neq u} \lambda_v \cdot v$ then

$$\lambda_s = \lambda \,//\, C_u^{1s\lambda} =$$

$$\lambda_u \left(1 + c_u \lambda_u \frac{e^{sc_\lambda} - 1}{c_\lambda}\right)^{-1} \left(e^{sc_\lambda} u - c_u \frac{e^{sc_\lambda} - 1}{c_\lambda} \sum_{v \neq u} \lambda_v v\right)$$

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6 If $\lambda = \sum \lambda_u u$ then $\operatorname{div}_u \lambda = c_u \lambda_u$

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7. We get $J(0) = 0$ &

$$\frac{dJ(s)}{ds} = J(s) // \ldots + \operatorname{div}_u \lambda_s$$

$$= c_u \lambda_u \left(1 + c_u \lambda_u \frac{e^{sc_\lambda} - 1}{c_\lambda}\right)^{-1} e^{sc_\lambda}$$

So $J(1) = \log \dfrac{c_\lambda + (e^{c_\lambda} - 1) c_u \lambda_u}{c_\lambda}$

which agrees with β-calculus:

$$\omega \left(1 + \frac{\left(-1 + e^{\alpha c_1 + \gamma c_2}\right) \alpha c_1}{\alpha c_1 + \gamma c_2}\right)$$

120802 Calculator.nb

In[1]:= $\int_0^t c_u \lambda_u \left(1 + c_u \lambda_u \frac{e^{s c_\lambda} - 1}{c_\lambda}\right)^{-1} E^{s c_\lambda} \, ds$

Out[1]= $-\mathrm{Log}[c_\lambda] + \mathrm{Log}\left[c_\lambda + \left(-1 + e^{t c_\lambda}\right) c_u \lambda_u\right]$

In[2]:= $\int_0^1 c_u \lambda_u \left(1 + c_u \lambda_u \frac{e^{s c_\lambda} - 1}{c_\lambda}\right)^{-1} E^{s c_\lambda} \, ds$

Out[2]= $-\mathrm{Log}[c_\lambda] + \mathrm{Log}\left[c_\lambda + \left(-1 + e^{c_\lambda}\right) c_u \lambda_u\right]$