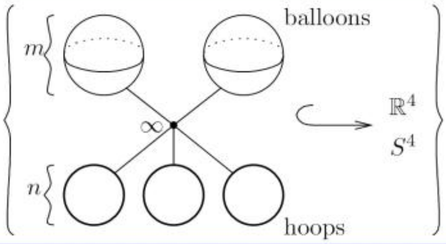
Balloons and Hoops and their Universal Finite–Type Invariant, BF Theory, and an Ultimate Alexander Invariant

Dror Bar–Natan in Hamburg, August 2012
http://www.math.toronto.edu/~drorbn/Talks/Hamburg-1208/

Abstract. Balloons are two-dimensional spheres. Hoops are one dimensional loops. Knotted Balloons and Hoops (KBH) in 4-space behave much like the first and second fundamental groups of a topological space - hoops can be composed like in π_1, balloons like in π_2, and hoops "act" on balloons as π_1 acts on π_2. We will observe that ordinary knots and tangles in 3-space map into KBH in 4-space and become amalgams of both balloons and hoops.
We give an ansatz for a tree and wheel (that is, free-Lie and cyclic word) -valued invariant Z of KBHs in terms of the said compositions and action and we explain its relationship with finite type invariants. We speculate that Z is a complete evaluation of the BF topological quantum field theory in 4D, though we are not sure what that means. We show that a certain "reduction and repackaging" of Z is an "ultimate Alexander invariant" that contains the Alexander polynomial (multivariable, if you wish), has extremely good composition properties, is evaluated in a topologically meaningful way, and is least-wasteful in a computational sense. If you believe in categorification, here's a wonderful playground.

$\mathcal{K}^{bh}(m,n)$.

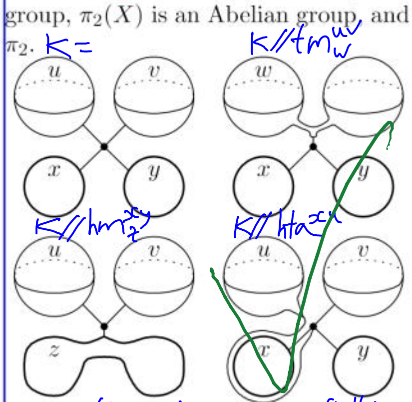


Example.

Teichner by Bergman

- u-Tangles inject into $\mathcal{K}^{bh}$.
- In fact, n-component v/w-tangles map into $\mathcal{K}^{bh}(n,n)$ and we have a conjectural understanding of $\mathcal{K}^{bh}$ in these terms.

Meta-Group-Action. If X is a space, $\pi_1(X)$ is a group, $\pi_2(X)$ is an Abelian group, and π_1 acts on π_2. $K=$



properties: 1. two associativities
2. two "meta-actions"
3. with $dm = \dots$, dm is "associative"

The Meta-Group-Action M. Let T be a set of "tail labels" ("balloon colours"), H a set of "head labels" ("hoop colours"), and let $\{h_x\}_{x\in H}$ be a set of formal symbols. Let $FL = FL(T)$ and $FA = FA(T)$ be the (graded) free Lie and free associative algebras on generators T and let $CW = CW(T)$ be the (graded) vector space of cyclic words on T, so there's $\mathrm{tr}: FA \to CW$, and both FA and CW are FL-modules. Let

$$M(T,H) := \left\{\mu = \left(\omega, \lambda = \sum_{x\in H} h_x\lambda_x\right) :\ \omega\in CW,\ \lambda_x\in FL\right\}$$

$$M(T,H) := \{\mu = (\omega, \bar\lambda = (\lambda_x)_{x\in H}) :\ \omega\in CW,\ \lambda_x\in FL\}$$

(e.g., $\mu = \dots$). With $\mu = (\omega, \lambda)$ define

$$tm_w^{uv} : \mu \mapsto \mu /. (u, v\mapsto w),$$

$$hm_z^{xy} : \mu \mapsto \left(\omega, \left(\dots, \widehat{\lambda_x}, \widehat{\lambda_y}, \dots, \mathrm{bch}(\lambda_x,\lambda_y)_z\right)\right)$$

$$hta^{xu} : \mu \mapsto \underbrace{\{\mu \mathbin{/\!\!/.} (u\mapsto e^{\mathrm{ad}\,\lambda_x}(\bar u)) / (\bar u \mapsto u)\}}_{\mu/\!\!/CC_u^{\lambda_x}} + \underbrace{(J_u(\lambda_x), 0)}_{\text{the "}J\text{-spice"}}$$

The ~~J-Spice~~ Meta-Cocycle J Let $J_u(\lambda) := J(1)$ where

$$J(0) = \mathrm{div}_u\,\lambda, \qquad \lambda_s = \cancel{CC_u^{s\lambda}(\lambda)},\ \lambda/\!\!/CC_u^{s\lambda}$$

o.k. ascertain.

$$\frac{dJ(s)}{ds} = (J(s) \mathbin{/\!\!/} \mathrm{der}(u\mapsto[\lambda_s, u])) + \mathrm{div}_u\,\lambda_s.$$

Claim

$$CC_u^{\mathrm{bch}(\lambda_1,\lambda_2)} = CC_u^{\lambda_1} /\!\!/ CC_u^{\lambda_2/\!\!/CC_u^{\lambda_1}}$$

$$J_u(\mathrm{bch}(\lambda_1,\lambda_2)) = J_u(\lambda_1)/\!\!/CC_u^{\lambda_2/\!\!/CC_u^{\lambda_1}} + J_u(\lambda_2/\!\!/CC_u^{\lambda_1})$$

and hence tm, hm, hta form a meta-group-action.

Also further annotations on July 27 version

Get beta-formulas from 2012-04/nb/Foundations.pdf √

Complete all Lie algebras! √